Brief Recollection

given $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ right exact

$A \in Ch(\mathcal{A})$ define (when we can)

$\mathbb{L}_n \mathcal{F}(A) = H_n(\mathcal{F}\, Tot^{\oplus} P_{\cdot\cdot})$

where $P_{\cdot\cdot} \to A_{\cdot}$ is an EM resolution

in general, if $C_{\cdot\cdot}$ is a double complex

we have spectral sequences

$\{{}^{I}E^{*}_{p,q}\}$ filtration by columns (p)

$\{{}^{II}E^{*}_{p,q}\}$ filtration by rows (p!)

${}^{II}E^{2}_{p,q} = H^{v}_{p} H^{h}_{q}(C)$ ← $C_{q,p}$

${}^{I}E^{2}_{p,q} = H^{h}_{p} H^{v}_{q}(C)$ ← $C_{p,q}$

converge to $H_{p+q}(Tot^{\oplus} C)$ when things are suff. bounded.

For computing $\mathbb{L}_n \mathcal{F}(A)$ w/ $P_{\cdot\cdot} \to A_\cdot$

$$H^v_q(\mathcal{F}P_{\cdot\cdot}) = L_q\mathcal{F}(A_*)$$

$${}^{I}E^2_{p,q} = H_p(L_q\mathcal{F}(A))$$

$$H^h_q(\mathcal{F}P_{\cdot\cdot}) = \mathcal{F}P^H_{q*}$$ res. of $H_q(A_\cdot)$

$$H^v_p(H^v_q\mathcal{F}P) = L_p\mathcal{F}(H_qA) = {}^{II}E^2_{p,q}$$

$$L_p\mathcal{F}(H_qA) \Rightarrow \mathbb{L}_{p+q}\mathcal{F}(A_\cdot) \quad \text{always}$$

$$H_p(L_q\mathcal{F}(A)) \Rightarrow \mathbb{L}_{p+q}\mathcal{F}(A) \quad \text{if } A \in Ch_{\geq 0}$$

Grothendieck Spectral sequence

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, "FG"'] \arrow[dr, "G"] & \\
\mathcal{C} & & \mathcal{B} \arrow[ll, "F"]
\end{tikzcd}$$

F, G left exact, enough injectives in $\mathcal{A}, \mathcal{B}$ and G takes injectives to F-acyclics

Reminder: $A \in \mathcal{B}$ is F-acyclic if $R^pF(A)=0$ if $p>0$

Then for any $A \in \mathcal{A}$, we have a convergent spectral seq.

$$E_2^{p,q} = (R^pF)\left((R^qG)(A)\right) \Rightarrow R^{p+q}(FG)(A)$$

Pf: Construction:

given $A \in \mathcal{A}$, choose inj. res $A \to I$.

apply G to get $GA \to GI$.

$\mathbb{R}^nF(GI)$

have $${}^{I}E_2^{p,q} \Rightarrow \mathbb{R}^nF(GI)$$
$$\parallel$$
$$H^p((R^qF)(GI))$$

GI's acyclic

$(R^qF)(GI)=0$ if $q>0$

$$\Rightarrow \mathbb{R}^nF(GI) = H^n(R^0F(GI))$$
$$H^n(FG(I)) = R^n(FG)(A)$$

$${}^{II}E_2^{p,q} = R^pF\big(\underbrace{H^q(GI)}_{\substack{\shortparallel \\ R^qG(A)}}\big) = R^pF(R^qG(A))$$

$${}^{II}E_2^{p,q} = R^pF(R^qG(A)) \Rightarrow \underset{\substack{\shortparallel \\ R^n(FG)(A)}}{R^nF(GI)}$$