Sheaf Cohomology

X topological space, $Ab(X)$ Cat of Abelian sheaves on X

a.k.a sheaves of Abelian groups on X.

given $f : X \longrightarrow Y$ continuous, can define

$f_* \mathcal{F}$ the sheaf on Y given by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$

This is a left exact functor $Ab(X) \longrightarrow Ab(Y)$

Can form $R^n f_* : Ab(X) \to Ab(Y)$

($Ab(X)$ has enough injectives)

Given X a top space $\pi^X : X \longrightarrow * = pt.$

$Ab(*) = Ab$

<u>Def</u> $H^n(X, \mathcal{F}) \equiv R^n \pi^X_*(\mathcal{F})$

$\pi^X_*(\mathcal{F}) = \mathcal{F}(X)$

deRham cohomology $\qquad X =$ smooth manifold (real)

Ω^n_X = sheaf of nth order diff. forms

$$\Omega^n_X(U) = \{ f \, dg_1 \wedge \dots \wedge dg_n \}$$

Properties: $df \wedge dg = -dg \wedge df$

$$d(fg) = f\,dg + g\,df$$

if x_i's loc. coords

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

$$\Omega^n_X(U) \xrightarrow{d} \Omega^{n+1}_X(U)$$

$$f \, dg_1 \wedge \dots \wedge dg_n \longmapsto df \wedge dg_1 \wedge \dots \wedge dg_n$$

Def $H^n_{dR}(X, \mathbb{R}) = H^n(\Omega^\bullet_X(X))$

Poincaré Lemma: If $U \subset \mathbb{R}^N$ is convex then

$$H^k_{dR}(U) = \begin{cases} 0 & \text{if } k \geq 1 \\ \mathbb{R} & \text{if } k = 0 \end{cases}$$

<u>Def</u> Constant sheaf $\underline{\mathbb{R}}$

defined as $\underline{\mathbb{R}}(U) = \{ f : U \to \mathbb{R} \mid f' = 0 \}$

locally constant.

Consider the cochain complex of sheaves

$$0 \to \underline{\mathbb{R}} \to C^\infty_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots$$

Poincaré lemma $\Rightarrow$ this is exact!

as a sequence of <u>sheaves</u>

i.e. it is locally exact.

$$\Omega^*_X : \underset{\Omega^0_X}{\overset{\shortparallel}{C^\infty_X}} \to \Omega^1_X \to \cdots$$

by def $H^n(\Omega^*_X(X)) = H^n_{dR}(X)$

$$\begin{array}{ccc} \Omega^\bullet_X \to I^{\bullet\bullet} & & \text{EM resolution.} \\ \uparrow & & \\ \underline{\mathbb{R}} & \longrightarrow & \operatorname{Tot} I^{\bullet\bullet} \text{ is an inj. res of } \underline{\mathbb{R}} \end{array}$$

$$\mathbb{H}^n(X, \Omega^\bullet_X) = \mathbb{R}^n \pi^X_*(\Omega^\bullet_X)$$

$$\| $$

$$H^n(\operatorname{Tot} I^{\bullet\bullet}(X))$$

$$\|$$

$$H^n(X, \underline{\mathbb{R}})$$

note $H^q(\Omega^\bullet_X) = 0$ because $\Omega^\bullet_X$ exact complex of sheaves

$q > 0$

$$H^p(X, H^q(\Omega^\bullet_X)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X)$$

$\|$ 0 unless $q = 0$

$\| \; H^{p+q}(X, \underline{\mathbb{R}})$

"collapse"!

$$H^n(X, \underline{\mathbb{R}})$$

$\| \; H^n(X, \underline{\mathbb{R}})$

$$H^n(X, \underline{\mathbb{R}} \quad)$$

$$H^p\left(H^q(X,\Omega^\bullet_X)\right) \Rightarrow \mathbb{H}^{p+q}(X,\Omega^\bullet_X)$$

$\Omega^\bullet_X$'s are "soft" sheaves $\Rightarrow H^q(X,\Omega^\bullet_X)=0$, $q>0$.

collapse!

$$H^n(\underbrace{H^0(X,\Omega^\bullet_X)}_{\Omega^\bullet_X(X)}) = \mathbb{H}^n(X,\Omega)$$

$$\mathbb{H}^n(X,\Omega) = H^n(X,\underline{\mathbb{R}})$$

$$H^n(\Omega^\bullet_X(X)) = H^n_{dR}(X) = H^n(X,\underline{\mathbb{R}})$$

X ~ Analytic manifold

$$0 \to \underline{\mathbb{C}} \to \Omega^0_{an} \xrightarrow{d} \Omega^1_{an} \to \cdots$$

$$\mathbb{H}^n(X,\Omega^\bullet_{an})$$

$$H^{p+q}(X,\underline{\underline{\mathbb{C}}}) = \mathbb{H}^{p+q}(X,\Omega)$$

$$H^p(X,H^q(\Omega^\bullet_X)) \Rightarrow \mathbb{H}^{p+q}(X,\Omega)$$

collapses by "analytic Poincaré" $\quad H^p(H^q(X,\Omega^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X,\Omega)$

deep fact "Hodge thy"

diffs are 0!
in spectral seq degs.

$$H^p(H^q(X,\Omega)) = H^q(X,\Omega^p)$$

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^q(X,\Omega^p)$$

"Hodge decomposition"

Same in Alg geom

$\mathbb{H}^n(X,\Omega_{alg})$ = alg deRham cohom.

in char p these are $\mathbb{F}_p$ vspaces.

$\mathbb{H}^n(X, W\Omega_{alg})$ mod for $W(\mathbb{F}_p) = \mathbb{Z}_p$

Serre "Geometrie Algebraique Geometrie Analytique"

if X smooth proj variety, $\mathcal{F}$ coherent $\Rightarrow$

$$H^n(X_{zar}, \mathcal{F}) = H^n(X_{an}, \mathcal{F})$$

Ω^n's are coherent

if X variety, $\mathcal{F}$ = constant

$$H^n(X_{zar}, \mathcal{F}) = 0, \quad n \neq 0.$$