Def An exact couple is a pair of objects $D, E \in \mathcal{A}$

& morphisms

$$\mathcal{E} : \left[\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} \right]$$

which is exact at each place

Define $d : E \to E$ by $d = jk$ then $d^2 = jkjk$
$= 0$

so can consider $H(E) = \dfrac{\ker d}{\operatorname{im} d}$
" E' "

Set $D' = i(D) = \operatorname{image}(i)$

Define $k' : E' \to D'$ $\quad k'(\bar{x}) = k(x)$ for $x \in \ker d$

$i' = i|_{D'}$ $\quad j'(x) = \overline{j(y)}$
$x \in D' = i(D)$ $\quad x = i(y)$

Lemma The above maps are well defined, and give a new exact couple

$$\mathcal{E}' = \left[\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd} \right]$$

Ex: If $C_\cdot$ is a complex w/ $F_\cdot$ filtration on C, have an exact seq. of chain complexes

$$0 \to F_{p-1}C \to F_pC \to F_pC/F_{p-1}C \to 0$$

giving a LES

$$\to H_n(F_{p-1}C) \to H_n(F_pC) \to H_n(F_pC/F_{p-1}C)$$

giving an exact Δ

$$\begin{tikzcd}[column sep=small]
\bigoplus_{p,n} H_n(F_pC) \arrow[rr] & & \bigoplus_{p,n} H_n(F_pC) \arrow[dl] \\
& \bigoplus_{p,n} H_n(F_pC/F_{p-1}C) \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\bigoplus_{p,q} H_{p+q}(F_pC) \arrow[rr, "{(1,-1)}"] & & \bigoplus_{p,q} H_{p+q}(F_pC) \arrow[dl, "{(0,0)}"] \\
& \bigoplus_{p,q} H_{p+q}(F_pC/F_{p-1}C) \arrow[ul, "{(-1,0)}"] &
\end{tikzcd}$$

...

$$\mathcal{E}: \quad \begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

$$\deg d = \deg k + \deg j$$

$$\mathcal{E}': \quad \begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

$$\deg k' = \deg k$$

$$\deg i' = \deg i$$

$$\deg j' = \deg j - \deg i$$

$$\deg d' = \deg k + \deg j - \deg i$$
$$= \deg d - \deg i$$

If we are given an exact couple w/ bidegrees

$$\begin{tikzcd} D \arrow[rr, "{(1,-1)}"] & & D \arrow[dl, "{(-a,a)}"] \\ & E \arrow[ul, "{(-1,0)}"] & \end{tikzcd}$$

defines a spectral sequence $\{E^r_{p,q}, d_r\}_{r \geq a}$ where

$$E^{a+r}_{p,q} = \left(E^{(r-1)}\right)_{p,q}$$

Aside

$\mathcal{A}$ ab cat, define $\mathcal{A}^{\mathbb{Z}^2}$ a new ab. cat w/ objects tuples $\{A_{p,q}\}_{p,q\in\mathbb{Z}}$

$$\mathrm{Hom}(\{A_{p,q}\},\{B_{p,q}\}) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Hom}_{i,j}(\{A_{p,q}\},\{B_{p,q}\})$$

where $\mathrm{Hom}_{i,j}(\{A_{p,q}\},\{B_{p,q}\})$

$$\| $$

$$\{f_{p,q} : A_{p,q} \to B_{p+i,q+j}\}$$

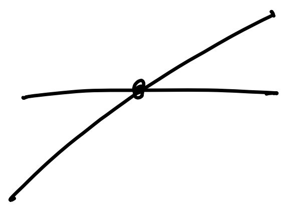
Given an exact couple as above

$$H_n = \varinjlim_{p} D_{p,n-p} \quad \text{w/r/ to the } i\text{'s}$$

$$\begin{tikzcd}
{} \arrow[r] & D_{p,n-p} \arrow[r, "i"] \arrow[dr, two heads] & D_{p+1,n-p-1} \arrow[r] & \cdots & H_n \\
& & F_pH_n \arrow[urr, hook] & &
\end{tikzcd}$$

<u>Goal:</u> "Homological dimension" "regularity – smoothness"

regularity has something to do with things being nicely cut out by correct # of eqns



hom dim = length of resolutions of modules (proj, inj, flat)

<u>Main result</u> hom dim $< \infty \iff$ ring is regular

max'l length of a minimal res = dim of ring (Krull)

Let M be a right R-module.

<u>Def</u> $pd(M)$ "projective dimension"

$= \min\{n \mid \exists\ 0 \to P_n \to \dots \to P_1 \to P_0 \to M \to 0\}$

P_i projective.

<u>Def</u> $id(M) = \min\{n \mid \exists\ 0 \to M \to I_0 \to \cdots \to I_n \to 0\}$
I_i injective.

<u>Def</u> $fd(M) = \min\{n \mid \exists\ 0 \to F_n \to \cdots \to F_0 \to M \to 0\}$
F_i flat.

<u>Theorem</u> (Global dim thm) the following are same:

1. $\sup\{id(M) \mid M \in \mathrm{Mod}_R\}$
2. $\sup\{pd(M) \mid M \in \mathrm{Mod}_R\}$
3. $\sup\{pd(R/I) \mid I \triangleleft_r R\}$
4. $\sup\{d \in \mathbb{Z}_{\geq 0} \mid \mathrm{Ext}^d_R(M,N) \neq 0 \text{ some } M,N\}$

the common value is called global dim of R

(right) $gd(R)$

projective dimension lemma (4.1.6)
TFAE for a R module M:

1. $pd(M) \leq d$
2. $\mathrm{Ext}^n_R(M,N) = 0$ if $n > d$ all N
3. $\mathrm{Ext}^{d+1}_R(M,N) = 0$ all N

4. if $0 \to M_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0$

where each P_i is projective, then

M_d also projective.

$3 \Rightarrow 4$ $\quad \mathrm{Ext}^{d+1}(M,N) = \mathrm{Ext}^1(M_d,N)$

$\Rightarrow M_d$ projective.