Things about Rings

Global dimension thm

The following #s are equal for any ring R

1. $\sup\{\operatorname{id}(M) \mid M \text{ an } R\text{-mod}\}$
2. $\sup\{\operatorname{pd}(M) \mid M \text{ an } R\text{-mod}\}$
3. $\sup\{\operatorname{pd}(R/I) \mid I \trianglelefteq R\}$
4. $\sup\{d \mid \operatorname{Ext}^d_R(M,N) \neq 0 \text{ some } M,N\}$

"global dim of R"

Tor dimension thm

The following #s are equal for any ring R

1. $\sup\{\operatorname{fd}(M) \mid M \text{ an right } R\text{-mod}\}$
2. $\sup\{\operatorname{fd}(M) \mid M \text{ an left } R\text{-mod}\}$
3. $\sup\{\operatorname{fd}(R/I) \mid I \trianglelefteq_\ell R\}$ 3½. $\{ \qquad \mid I \trianglelefteq_r R\}$
4. $\sup\{d \mid \operatorname{Tor}^d_R(M,N) \neq 0 \text{ some } M,N\}$

" Tor dim of R "

Prop (4.1.5) If R is right Noeth then

- $fd(M) = pd(M)$ all f.g. right R-mods M
- $\text{Tor dim}(R) = (\text{right})\ \text{glob}(R)$

Examples

Def A ring R is called (right) semisimple if R is a direct sum of simple right R-mods.

ex: $M_n(F) = \left\{ \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} 0 \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \oplus \cdots$

Wedderburn's thm: R semisimple $\iff$

$R = M_{n_1}(R_1) \times \cdots \times M_{n_m}(R_m)$

R_i division ring.

Thm
TFAE

1. R semisimple
2. R has (left & right) glob. dim 0
3. every R-mod is proj

4. every R-mod is injective
5. R Noeth & every R-mod is flat
6. R Noeth & of dim 0

Def A ring R is quasi-Frobenius if it's r & l Noeth & R is injective as a (r & l) R-mod.

Thm TFAE
1. R quasi-Frob.
2. every proj right R-mod is injective   2') ... left
3. " inj left - - proj   3') - - right

Def An algebra R over a field k is Frobenius if $R \simeq Hom_k(R,k)$ as a right R-module.

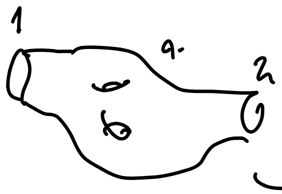
$1 \longrightarrow \theta$

injective R-module.

2 dim'l Topological Quantum Field Theories
(1+1)

1 dim'l oriented mflds w/ out boundaries ⇝ Vector spaces.

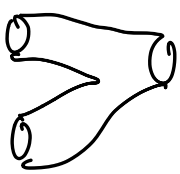
 ⇝ $V \otimes V$

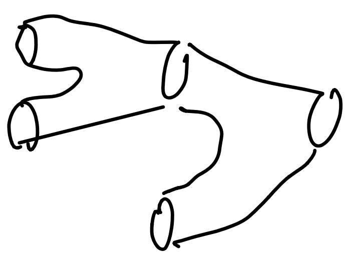
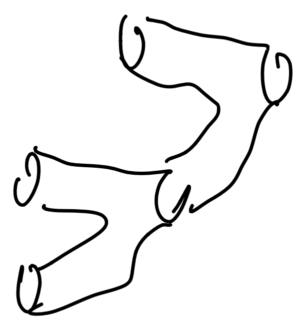


$\longrightarrow V_1 \xrightarrow{f} V_2$

 $\longrightarrow V$

 $k \longrightarrow V$

 $V \otimes V \longrightarrow V$

 $\simeq$ 

 $V \xrightarrow[\theta]{} k$

Def A comm. ring R is called Gorenstein if $id(R)$ is finite. In this case, $id(R) =$ Krull $\dim(R)$

q. frob $\Rightarrow$ Gorenstein.

q. frob = Gorenstein, dim 0.

Def. A ring R is called (right) heriditary if every right ideal is projective. A commutative hereditary integral domain is also called a "Dedekind domain."

Thm R is r. heriditary iff r. gl dim $(R) \leq 1$.

Def Krull dim (R).

Def Let R be a comm. local ring, max'l ideal $m \trianglelefteq R$, $k = R/m$. Then m/m^2 is vector space over k

$$\text{emb.}\dim(R) = \dim_k (m/m^2)$$

Def A local ring R is called regular if emb dim = Krull dim.

Krull $\leq$ emb. dim.

If M is a f.g. R-module, we say $(x_1, \ldots, x_n) \in R^n$ is a regular seq. on M (aka an M-sequence) if

- x_1 a nonzero div on M
- x_2 - - - on M/x_1M
- $\vdots$
- x_i - - - $M/(x_1, \ldots, x_{i-1})M$

$G(M)$ = length of largest M-sequence

$G(R) \leq$ Krull dim(R) any R local

Def R is Cohen-Macaulay if $G(R) =$ Krull dim(R)

Thm (Main theorem 4.4.16)

A local ring R is regular iff gl dim$(R) < \infty$

and in this case

$G(R) =$ Krull dim$(R) =$ emb. dim$(R) =$ p.dim$(R) =$ gl.dim(R)