Correction: (last lecture)

Exercise: if we consider SES_R a subcat of $Ch(Mod_R)$ then the smallest Abelian subcat of $Ch(Mod_R)$ containing SES_R & containing all objects of $Ch(-)$ iso. to its objects is $Ch(\ )$.

the category of bounded (eventually 0 for $i \gg 0$ or $i \ll 0$) complexes

Next week: entirely virtual.

Def: if $A_\bullet, B_\bullet \in Ch_R$, we say that a chain map $f_\bullet : A_\bullet \to B_\bullet$ is a quasi-isomorphism if the induced maps $H_n(f_\bullet) : H_n(A_\bullet) \to H_n(B_\bullet)$ are isoms.

example:

consider a presentation of the R-module

M $\qquad R = k[x,y] \qquad M = \dfrac{k[x,y]}{(x,y)} = \text{"}k\text{"}$

$$0 \to R \to R^2 \to R \to M \to 0$$
$$1 \longmapsto \bar{1}$$

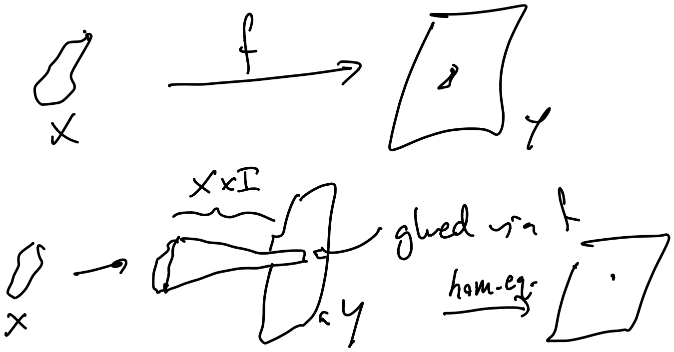
can read this as

$$\begin{array}{ccccccccc} 0 & \to & R & \to & R^2 & \to & R & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & 0 & \to & 0 & \to & M & \to & 0 \end{array}$$

is quasi-isom.

eventually we'll talk a lot more about quasi-isoms & categorical interpretation 'Derived Category'

From the point of view of q. isoms, injectivity & surjectivity lose meaning.



On the other hand, inj./surj. on the complex level (not q.iso) is very important.

$$\text{SES} \longrightarrow \text{LES}.$$

<u>Lemma</u> (snake lemma)

(alternate version)

$$\begin{tikzcd}
\cancel{0} \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \cancel{0}
\end{tikzcd}$$

map of SES's induces an exact sequence

$$\cancel{0 \to} \ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \cancel{\to 0}$$

Exercise 1.3.3 (5-Lemma)

<u>Theorem (1.3.1)</u>

Given a SES_{Ch_R} i.e.

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

we get an exact sequence of homology groups

$$\cdots \to H_{n+1}(C) \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \cdots$$

$$\begin{tikzcd}
 & H_*(A) \arrow[dr, "0"] & \\
H_*(C) \arrow[ur, "-1"] & & H_*(B) \arrow[ll, "0"]
\end{tikzcd}$$

$$H_*(A) = \bigoplus H_i(A)$$

gives a functor

$$SES_{Ch_R} \longrightarrow Ch_R$$

landing in exact complexes.

<u>Pf</u>: consider

$$\begin{array}{ccccccccc}
0 & \to & A_n & \to & B_n & \to & C_n & \to & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \to & A_{n-1} & \to & B_{n-1} & \to & C_{n-1} & \to & 0
\end{array}$$

kernels are $Z_n(A)$, $Z_n(B_n)$, $Z_n(C_n)$

cokernels are $A_{n-1}/\partial A_n, \ldots$

$$\begin{tikzcd}
 & \frac{A_n}{\partial A_{n+1}} \arrow[r] \arrow[d, "d"] & \frac{B_n}{\partial B_{n+1}} \arrow[r] \arrow[d, "d"] & \frac{C_n}{d C_{n+1}} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}A \arrow[r] & Z_{n-1}B \arrow[r] & Z_{n-1}C &
\end{tikzcd}$$

snake for this gives LES.

Functoriality follows from functoriality of snake.

✓ □.

Chain Homotopies

<u>Motivating idea</u>: when is there a "good reason" for two maps $A_\bullet \underset{g}{\overset{f}{\rightrightarrows}} B_\bullet$ to be induce the same map in homology?

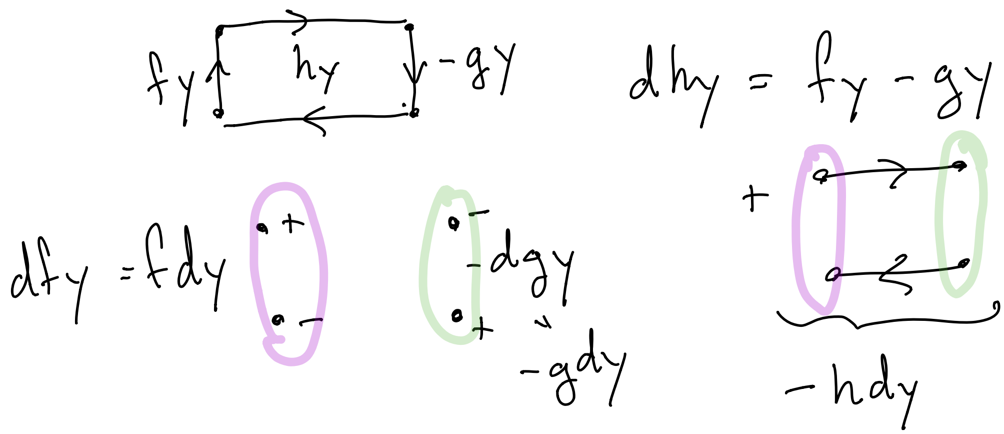
to demonstrate homology same, want for each $z \in Z_n A$, $fz = gz + dx$

$x = x(z)$

would like to "promote this" to a map

$h: A_n \to B_{n+1}$ s.t. if $z \in Z_n A$

$y \in A_n$ $\qquad dh(z) = f(z) - g(z)$.



Def A chain homotopy $h: f \to g$ between two chain maps $f, g: A_\bullet \to B_\bullet$ is a collection of R-mod maps $h_i: A_i \to B_{i+1}$ s.t.

$$f - g = dh + hd.$$

Prop: if $\exists$ a chain homotopy $h: f \to g$ then $H_n(f) = H_n(g)$, all n.

Def: f is called null-homotopic if f is chain hom. eq. to 0.

Rem null homotopic $\Rightarrow H(f) = 0$.

Converse?

Exercise identity is null homotopic $\Longleftrightarrow$ complex is "split exact"

Def A complex is split, if $\exists$ maps $s : A_n \to A_{n+1}$

$$\begin{tikzcd} A_n \arrow[r, "d"'] & A_{n-1} \arrow[l, bend right, "s"'] \arrow[r, "d"'] & A_{n-2} \arrow[l, bend right, "s"'] \end{tikzcd} \qquad \text{s.t. } d = dsd$$

$$(\Rightarrow (ds)^2 = ds$$

s is called a splitting.

(end at 1.4)