One purpose of homological algebra:

repair failure of exactness

$A \xrightarrow{f} B$ hom. of ab gps.

$A/2A \xrightarrow{f/2} B/2B$ induced map.

if f is surjective, so is $f/2$.

if f is injective, $f/2$ may or may not be.

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightsquigarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow[\;]{\underset{0}{\overset{\cdot 2/2}{\shortparallel}}} \mathbb{Z}/2\mathbb{Z}$$

Step 1: construct exact sequence (w/ map of interest)

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

Step 2: construct compatible resolutions

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "1"] \arrow[d, "{(-1,2)}"] & \mathbb{Z} \arrow[d, "2"] & \\
\mathbb{Z} \arrow[r, "i_1"] \arrow[d] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\pi_2"] \arrow[d, "{(2,1)}"] & \mathbb{Z} \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

rows are split!

Note: if we had a split injection (or a split SES) of Ab. gps

$$0 \to A \to A \oplus B \to B \to 0$$

$$A \oplus B / 2(A \oplus B) = A/2A \oplus B/2B$$

Apply mod 2 to diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \underset{\mathrm{Tor}^1(\mathbb{Z},\mathbb{Z}/2)}{0} \arrow[r] \arrow[d] & \underset{\mathrm{Tor}^1(\mathbb{Z}/2,\mathbb{Z}/2)}{\mathbb{Z}/2} \arrow[d] \arrow[ddll, bend left, out=0, in=180] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "1"] \arrow[d, "{(-1,2)}"] & \mathbb{Z}/2 \arrow[d, "2"] \\
\mathbb{Z}/2 \arrow[r, "i_1"] \arrow[d] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r, "\pi_2"] \arrow[d, "{(2,1)}"] & \mathbb{Z}/2 \arrow[d] \\
\mathbb{Z}/2 \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0
\end{tikzcd}$$

$\mathbb{Z} \otimes \mathbb{Z}/2 = \mathrm{Tor}^0(\mathbb{Z},\mathbb{Z}/2)$, $\mathbb{Z} \otimes \mathbb{Z}/2 = \mathrm{Tor}^0(\mathbb{Z},\mathbb{Z}/2)$, $\mathbb{Z}/2 \otimes \mathbb{Z}/2 = \mathrm{Tor}^0(\mathbb{Z}/2,\mathbb{Z}/2)$

snake lemma "explains" failure of left exactness

$0 \to A \to B \to C \to 0$

$\rightsquigarrow$ resolutions

$A_\bullet$ is a resolution of A ("presentation")

$B_\bullet$ B

$$\underbrace{\to A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0}_{A_\bullet} \to A \to 0 \quad \text{exact complex}$$

$\rightsquigarrow$ SES of complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$

special in that e.g. each row is split.

$\Rightarrow$ apply (troublesome) functor F $(\otimes \mathbb{Z}/2)$

$0 \to FA_\bullet \to FB_\bullet \to FC_\bullet \to 0$ w/ assoc LES.

at 0 level (H^0) get back $FA \to FB \to FC \to 0$

if F is r. exact.

Other functors

$$A \longmapsto A/2A = A \otimes_{\mathbb{Z}} \mathbb{Z}/2$$

$$M \longmapsto M \otimes_R N \qquad \text{given } N \text{ a left } R\text{-mod}$$

$$- \otimes_R N : \mathrm{Mod}_R \longrightarrow \mathrm{Ab}$$

Def N is called a flat R-module if $\otimes_R N$ is exact.

G-modules

Def G group, we say M is a G-module if it is an Ab.gp w/ action of G i.e.

$$G \times M \longrightarrow M \qquad g(m+n) = gm + gn$$
$$(g,m) \longrightarrow gm$$
$$(gh)m = g(hm)$$
$$1 \cdot m = m$$

Remark G-modules $\longleftrightarrow$ $\mathbb{Z}G$ modules

Def if M is a G-module
$$M^G = \{ m \in M \mid gm = m \text{ all } g \in G \}$$

$$M_G = M / \langle m - gm \rangle_{g \in G}$$

- ^{G} is left exact
- $_G$ is right exact.

We want a machine which takes a additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between Abelian categories which is left (or right) exact, and produces a seq. of supplementary functors which extend the left (or right) exact seq. to a LES.

$R^i F : \mathcal{A} \longrightarrow \mathcal{B}$ $\qquad$ $R^0 F = F$

$$0 \to A \to B \to C \to 0$$

$$\rightsquigarrow$$

$$\cdots \to R^1 F C \to R^0 F A \to R^0 F B \to R^0 F C \to 0$$

Derived Functors

δ-Functors

Informally a δ-functor is a procedure for turning SES to LES.

<u>Def</u> $\mathcal{A}, \mathcal{B}$ Abelian Categories a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ is a collection of additive functors $T_n : \mathcal{A} \to \mathcal{B}$ $n \geq 0$ together w/, for every SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $SES_{\mathcal{A}}$ a morphism $T_n C \xrightarrow{\delta_n} T_{n-1} A$ such that we get a long exact sequence (exact complex)

$$\cdots \to T_n(A) \xrightarrow{T_n(f)} T_n(B) \xrightarrow{T_n(g)} T_n(C) \to T_{n-1}(A) \xrightarrow{T_{n-1}(f)} \cdots$$

$$\cdots \to T_0 B \to T_0 C \to 0.$$

and such that δ_n are natural in the sense that whenever we have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

we get a comm. diagram

$$\begin{tikzcd} T_n(C) \arrow[r, "\delta"] \arrow[d, "T_n\gamma"'] & T_{n-1}(A) \arrow[d, "T_{n-1}\alpha"] \\ T_n(C') \arrow[r, "\delta"] & T_{n-1}(A') \end{tikzcd}$$

Example (2.1.2)

$H_\bullet$ is a δ-functor from $Ch_{\mathcal{A}}$ to $\mathcal{A}$

Ex (2.1.3)

A an Ab.gp, set $T_0(A) = A/nA$

$T_1(A) = \{a \in A \mid na = 0\} = A[n]$

Claim: if $0 \to A \to B \to C \to 0$ exact.

then get a LES:

$$0 \to A[n] \to B[n] \to C[n] \to A/nA \to B/nB \to C/nC \to 0$$

How?

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\cdot n"] & B \arrow[r] \arrow[d, "\cdot n"] & C \arrow[r] \arrow[d, "\cdot n"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

given S, T δ-functors from $\mathcal{A}$ to $\mathcal{B}$

a morphism of δ-functors $\varphi: S \to T$

is a collection of natural transformations

$\varphi_i : S_i \to T_i$ s.t. for $0 \to A \to B \to C \to 0$ in $SES_{\mathcal{A}}$

we get a commutative ladder:

$$\begin{tikzcd} \arrow[r] & T_n(A) \arrow[r] \arrow[d, "\varphi_n(A)"] & T_n(B) \arrow[r] \arrow[d, "\varphi_n(B)"] & T_n(C) \arrow[r] \arrow[d] & T_{n-1}(A) \arrow[r] \arrow[d] & {} \\ \arrow[r] & S_n(A) \arrow[r] & S_n(B) \arrow[r] & S_n(C) \arrow[r] & S_{n-1}(A) \arrow[r] & {} \end{tikzcd}$$

<u>Def</u> a δ-functor T from $\mathcal{A}$ to $\mathcal{B}$ is universal if $\forall$ δ-functors $S: \mathcal{A} \to \mathcal{B}$ $\exists!$ all nat. transformations

$f_0 : S_0 \longrightarrow T_0$ $\exists$ unique $f_0 : S \longrightarrow T$ extending it.

if $\exists$ univ. δ functor extending T_0

we say that the T_i's are the

"satellite functors for T_0"