Given an Abelian category how to prove 5-lemma?

$$\begin{tikzcd}
A' \arrow[r, "f"] \arrow[d, "a"'] & B' \arrow[r] \arrow[d, "b"'] & C' \arrow[r] \arrow[d, "c"'] & D' \arrow[r] \arrow[d, "d"'] & E' \arrow[d, "e"'] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

if b, d monic, a epic want to show c is monic

Fun proof by diagram chase. (elements of eg. Ab gps)

Consider monic in an Ab. cat. $C' \xrightarrow{c} C$

standard def in Ab gps: $\forall x \in C'$, $c(x) = 0 \Rightarrow x = 0$

In a general Ab cat if the composition $X \xrightarrow{f} C' \xrightarrow{c} C$ is zero for some f, then $f = 0$. $cf = 0 \Rightarrow f = 0$.

Def $C(X) = \mathrm{Hom}_{\mathcal{A}}(X, C)$ are called the (Ab gp) of X-valued elements of C.

note, if we have any morphism $A \xrightarrow{g} B$ then we have a natural gp hom

$A(X) \xrightarrow{g(X)} B(X)$ by precomposition - i.e.

$$\begin{tikzcd} & A \arrow[r, "g"] & B \\ X \arrow[ur, "\varphi"] \arrow[urr, bend right, "g\varphi"'] & & \end{tikzcd} \qquad \begin{array}{c} \varphi \longmapsto g\varphi \\ A(X) \longrightarrow B(X) \end{array}$$

So, this actually gives a functor (Yoneda "additive version")

$$\mathcal{A} \xrightarrow{h} \mathrm{Fun}_{Add}(\mathcal{A}^{opp}, Ab)$$

$$C \longmapsto (X \overset{h_C}{\longmapsto} \mathrm{Hom}_{\mathcal{A}}(X, C))$$

i.e. $C \longmapsto (X \longmapsto C(X))$

$X \xrightarrow{f} Y$ gives $C(Y) \to C(X)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed] & Y \arrow[d] \\ & C \end{tikzcd}$$

Fundamental theorem of Category theory
(Yoneda Lemma / Additive version)

$$h : \mathcal{A} \longrightarrow \mathrm{Fun}_{add}(\mathcal{A}^{op}, Ab)$$
$$C \longmapsto C(\ )$$

is a fully faithful exact functor between Abelian Categories

Pf: Exercise.

Remarks

$A \longrightarrow B$ is monic/epic iff

Yoneda $h_A \longrightarrow h_B$ is monic/epic iff

def $A(X) \longrightarrow B(X)$ is monic/epic $\forall X$

notice - this is a map of Ab gps.

Consequently: to check the 5 lemma above for a general Ab. cat, i.e. to show

$C' \xrightarrow{c} C$ monic it suffices to show

$C'(X) \xrightarrow{c(X)} C(X)$ is monic $\forall X \in \mathrm{ob}\,\mathcal{A}$.

But, can check this by applying $\mathrm{Hom}(X,-)$ to the entire diagram!

As noted, a map is epic/monic $\Leftrightarrow$ stays after applying $\mathrm{Hom}(X,-)$ $\forall X$

$\Rightarrow$ we get the diagram $\quad \mathrm{Hom}(X,-)$
$$A \mapsto A(X)$$

$$\begin{tikzcd}
A'(X) \arrow[r] \arrow[d] & B'(X) \arrow[r] \arrow[d, hook] & C'(X) \arrow[r] \arrow[d, hook, "?"] & D'(X) \arrow[r] \arrow[d, hook] & E'(X) \arrow[d] \\
A(X) \arrow[r] & B(X) \arrow[r] & C(X) \arrow[r] & D(x) \arrow[r] & E(X)
\end{tikzcd}$$

$\forall X$ works

$\Longrightarrow$ monic by def of monic.

yes — these are ab gps, so can choose elements as before!

Exercise *

Suppose $\mathcal{A}$ is an Abelian category, X an object, $R = \mathrm{Hom}_{\mathcal{A}}(X, X) = \mathrm{End}_{\mathcal{A}}(X)$

Show that R is an associative unital algebra, and

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(X, A) = A(X) \\ \mathcal{A} & & Ab \end{array}$$

can be regarded as an additive functor

$$\mathcal{A} \longrightarrow Mod_R$$ which is exact.