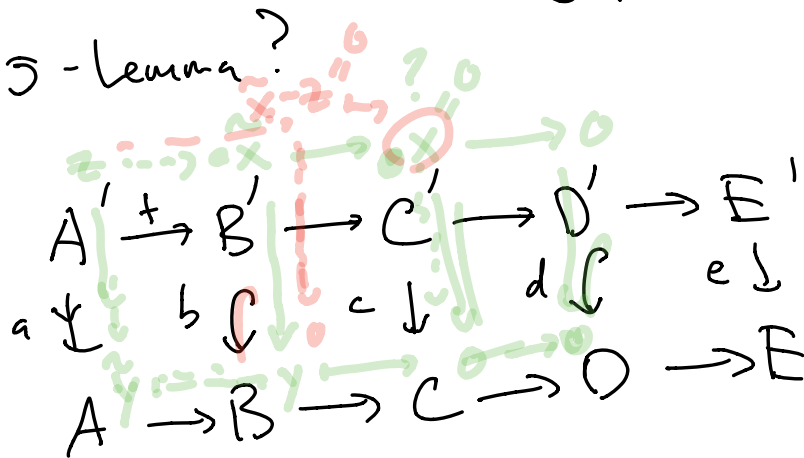


Given an Abelian category how to prove
 \mathfrak{S} -lemma?



if b, d monic, a epic want to show
 c is monic

For proof by diagram chase. (elements of
 eg. Ab grps)

Consider monic in an Ab. cat. $C' \xrightarrow{c} C$

standard def in Ab grps: $\forall x \in C', c(x) = 0 \Rightarrow x = 0$

In a general Ab cat if the composition

$$X \xrightarrow{f} C' \xrightarrow{c} C$$

is zero for some f , then

$$f = 0. \quad cf = 0 \Rightarrow f = 0.$$

Def $\mathcal{C}(X) = \text{Hom}_X(X, \mathcal{C})$ are called the $(A \downarrow \text{gp})$
of X -valued elements of \mathcal{C} .

note, if we have any morphism $A \xrightarrow{g} B$
then we have a natural gp hom

$$A(X) \xrightarrow{g(X)} B(X) \text{ by precomposition - i.e.}$$

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & A \xrightarrow{g} B \\ & \searrow & \nearrow \\ & & g\varphi \end{array} \quad \begin{array}{ccc} \varphi & \mapsto & g\varphi \\ A(X) & \mapsto & B(X) \end{array}$$

So, this actually gives a functor (Yoneda "additive version")

$$\mathcal{A} \xrightarrow{h} \text{Fun}_{\text{Add}}(\mathcal{A}^{\text{op}}, \mathcal{A} \downarrow \text{gp})$$

$$\mathcal{C} \longmapsto (X \longmapsto \text{Hom}_X(X, \mathcal{C}))$$

i.e. $\mathcal{C} \longmapsto (X \longmapsto \mathcal{C}(X))$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow \\ & & \mathcal{C} \end{array} \text{ gives } \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$$

Fundamental theorem of Category theory (Yoneda Lemma / Additive version)

$$h: A \rightarrow \text{Fun}_{\text{add}}(A^{\text{op}}, \text{Ab})$$

$$C \rightarrow \text{CC}$$

is a fully faithful exact functor between
Abelian Categories

Pf Exercise.

Remarks

$A \rightarrow B$ is monic/epic iff

Yoneda $h_A \rightarrow h_B$ is monic/epic iff

def $A(X) \rightarrow B(X)$ is monic/epic $\forall X$

note - this is a map of Ab grps.

Consequently i to check the 5 lemmas above

for a general Ab-cat, i.e. to show

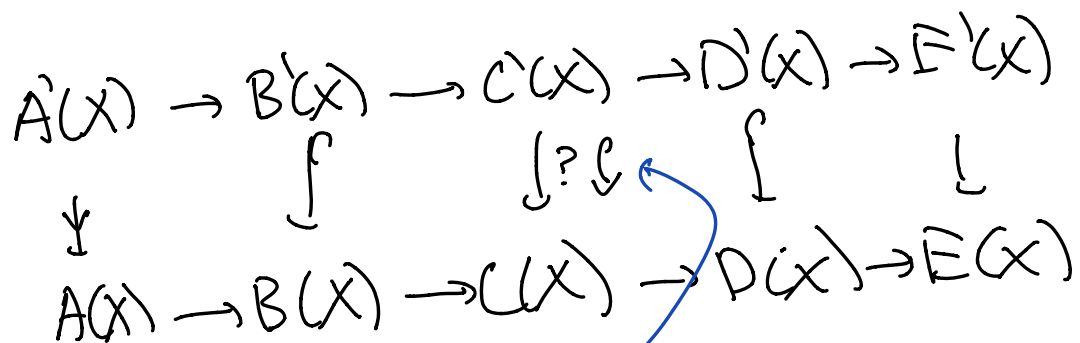
$C' \xrightarrow{c} C$ monic it suffices to show

$C'(X) \xrightarrow{c(X)} C(X)$ is monic $\forall X \in \text{ob } \mathcal{A}$.

But, can check this by applying $\text{Hom}(X, -)$ to the entire diagram!

As noted, a map is epic/monic \iff stays after apply $\text{Hom}(X, -)$

\implies we get the diagram $\text{Hom}(X, -)$
 $A \rightarrow A(X)$



$\forall X$ works

\implies monic by def of monic.

yes - these are all gps, so can check elements as before!

Exercise *

Suppose \mathcal{A} is an Abelian category,

X an object, $R = \text{Hom}_{\mathcal{A}}(X, X) = \text{End}_{\mathcal{A}}(X)$

Show that R is an associative unital algebra,

and $\mathcal{A} \longrightarrow \text{Hom}_{\mathcal{A}}(X, \mathcal{A}) = \mathcal{A}(X)$

$\mathcal{A} \quad \mathcal{A}b$

can be regarded as an additive functor

$\mathcal{A} \longrightarrow \text{Mod}_R$ which is exact.