In videos, showed that if we have an exact δ-functor

$\{T_i\}: \mathcal{A} \to \mathcal{B}$ s.t. each T_i, $i>0$

is coeffacable, then $\{T_i\}$ is universal,

and so all the "satellite functors" T_i, $i>0$

are uniquely determined by T_0.

Do they exist? Constructions via derived functors.

Basic procedure:

Given our right exact functor F, an object A,

Choose a exact complex $\to P_2 \to P_1 \to P_0 \to A \to 0$

giving a q-isom $P_\bullet \to A$

where each P_i is a projective object of $\mathcal{A}$.

Then we define $F_i(A) = H_i(FP_\bullet)$

homology

$$FP_{i+1} \to \boxed{FP_i} \to FP_{i-1}$$

Step 1: Projective objects

Lemma Let $\mathcal{A}$ be an Abelian category, $P \in \mathcal{A}$ then TFAE:

1) $\mathrm{Hom}(P, -)$ is right exact
2) Every SES $0 \to A \to B \to P \to 0$ splits
3) Any epic map $A \to P$ splits
4) given any diagram

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

w/ bottom row exact, $\exists$ a map $P \to B$ s.t. diagram commutes

Pf $(3 \Rightarrow 4)$

Suppose 3 holds, consider

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"'] & B \arrow[r] & 0 \end{tikzcd}$$

Define $A \times_B P = \ker\left(A \times P \xrightarrow{g\pi_1 - f\pi_2} B\right)$

note, by construction, have a comm. diagram

$$\begin{tikzcd} A \times_B P \arrow[r, "\pi_2"] \arrow[d] & P \arrow[d, "f"] \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

Claim: π_2 is epic

let $X \in \mathcal{A}$, let $x \in P(X)$. Set $\bar{x} \in B(X)$ its image. since g is epic, $\exists\, y \in A(X)$ mapping to $\bar{x}$. Then $(x, -y) \in (A \times_B P)(X)$ and $\pi_2(x, -y) = x$.

By hypothesis, $A \times_B P \to P$ has a splitting and now use:

$$\begin{tikzcd} A \times_B P \arrow[r] \arrow[d] & P \arrow[d] \arrow[l, bend right, dashed] \arrow[dl, bend right] \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

<u>Def</u> $P \in \mathcal{A}$ is called projective if the equiv. conditions of the Lemma hold.

Main point: additive functors always preserve split exact sequences.

If P is projective, given a r. exact $F_0 = F$

suppose we can find $G \twoheadrightarrow P$ w/ $F_1 G = 0$

$$0 \to K \to G \to P \to 0 \quad \text{splits}$$

$\Rightarrow$ LES for $\{F_i\}$ is actually a series of SESs.

$$F_i K \oplus F_i P = F_i G$$

Lemma: additive functors preserve split short exact sequences.

Pf. $A = B_1 \oplus B_2$ means we have

$$\begin{tikzcd} A \arrow[r, shift left, "\pi_i"] & B_i \arrow[l, shift left, "\iota_i"] \end{tikzcd} \quad \text{s.t. } \pi_i \iota_i = id_{B_i}$$

$$\iota_1 \pi_1 + \iota_2 \pi_2 = id_A$$

$\square$

Projective objects

Def an Ab. cat $\mathcal{A}$ has enough projectives if $\forall A \in \mathcal{A}$, $\exists P \twoheadrightarrow A$ w/ P projective.

ex: Mod_R has enough projectives.

Free modules are projective!

$$\begin{tikzcd} & R \arrow[d] \arrow[dl, dashed] & 1 \arrow[d, mapsto] \\ N \arrow[r, two heads] & M \arrow[r] & 0 \end{tikzcd} \qquad n \leftarrow\!\shortmid 1, \quad 1 \mapsto m, \quad n \mapsto m$$

$\mathrm{Hom}_R(R, M) = M$

easy to check: $\oplus$'s of projectives are projective.

Mod_R has enough projectives because any module has a generating set. $R^n \twoheadrightarrow M$

Summand of projectives are projective

$$\begin{tikzcd} & Q_1 \oplus Q_2 = P \arrow[dl, bend right] \arrow[d] \arrow[d, dashed, shift left=2, "0"] & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Prop: An R-module M is proj. $\Leftrightarrow$

$M \oplus Q = R^I$ is free for some Q

Pf: if M is proj, consider a surjection

$R^I \twoheadrightarrow M$, which splits. D.

Ex: if R a PID, M f.g. proj $\Rightarrow$ free.

if R is division, M proj $\Rightarrow$ free.

if R is a comm. loc. ring. proj $\Rightarrow$ free.

$$R = \mathbb{Z}/6\mathbb{Z} \qquad P = (\mathbb{Z}/6\mathbb{Z})/(2\mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

$$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$ (with "=" between $\mathbb{Z}/6\mathbb{Z}$ and this) is proj, not free.

$\mathbb{Z}[\sqrt{-5}]$ has nonfree projectives.

Def A resolution of an object $A \in \mathcal{A}$ is an complex $P_\bullet$ w/ a map $P_0 \to A$.
indx ≥ 0

s.t. $\to P_2 \to P_1 \to P_0 \to A \to 0$

is exact.

it is a projective resolution if all P_i's are projective.

Two handy lemmas:

Lem 1 if $A \xrightarrow{f} B$ in $\mathcal{A}$ and $P_\bullet, Q_\bullet$ are proj. res'ns for A & B resp. then $\exists$

$f_\bullet : P_\bullet \to Q_\bullet$ s.t.

$$\begin{array}{ccccc} P_1 & \to & P_0 & \to & A \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\ Q_1 & \to & Q_0 & \to & B \end{array}$$ commutes

and further, $f_\bullet$ is unique up to homotopy equivalence.

Lem 2 if $0 \to A' \to A \to A'' \to 0$ is SES.

and $P_\bullet'$ & $P_\bullet''$ are proj. resolutions of A', A''

then can find a proj. res $P_\bullet$ of A w/ each $P_i \simeq P_i' \oplus P_i''$, giving a SES of complexes

$$0 \to P_\bullet' \to P_\bullet \to P_\bullet'' \to 0 \quad \text{(split in each } i\text{)}$$