

Notation if A_\bullet is a resolution of A

then we'll write (today) A_\bullet^+

for complex $\dots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow A \rightarrow 0$

Lemma Given an abelian cat \mathcal{A} , objects

A, B , $f: A \rightarrow B$, projective resolutions

A_\bullet, B_\bullet of A & B then $\exists f_\bullet: A_\bullet \rightarrow B_\bullet$

s.t. extends the map $A \rightarrow B$ via

a map $A_\bullet^+ \rightarrow B_\bullet^+$. Further, f_\bullet is unique up to homotopy.

Pf:

$$\begin{array}{ccccccc} A_1 & \rightarrow & A_0 & \rightarrow & A & \rightarrow & 0 \\ & & \vdots & & \searrow & \downarrow & \\ & & B_1 & \rightarrow & B_0 & \rightarrow & B \rightarrow 0 \end{array}$$

exists since A_0 proj, $B_0 \twoheadrightarrow B$.

gives

$$\begin{array}{ccccc}
 & & A_1 & \longrightarrow & A_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & \swarrow h_1 & \downarrow [f_1, g_1] & & \downarrow [f_0, g_0] & & \downarrow f & & \\
 & & B_1 & \longrightarrow & B_0 & \longrightarrow & B & \longrightarrow & 0
 \end{array}$$

\nearrow proj. \searrow \rightarrow ind.

keep going. D.

Remark: seq. on bottom didn't need to be proj. top didn't need to be exact?

Ex confirm or deny \rightarrow

Lemma (Horseshoe)

Given an exact seq $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$

in A , proj. res. $A_0 \leq C$ of $A \leq C$,

can find proj. res B_0 of B st.

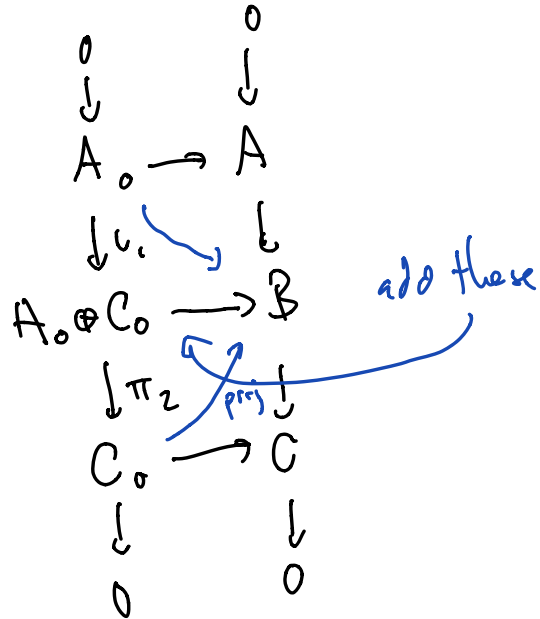
have an exact seq $0 \rightarrow A_0^+ \rightarrow B_0^+ \rightarrow C_0^+ \rightarrow 0$

which satisfies the condition that

for each i , $0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$
 is split exact. $i \geq 0$.

"Pf" Set $B_i = A_i \otimes C_i$

define $B_0 \rightarrow B$



general step

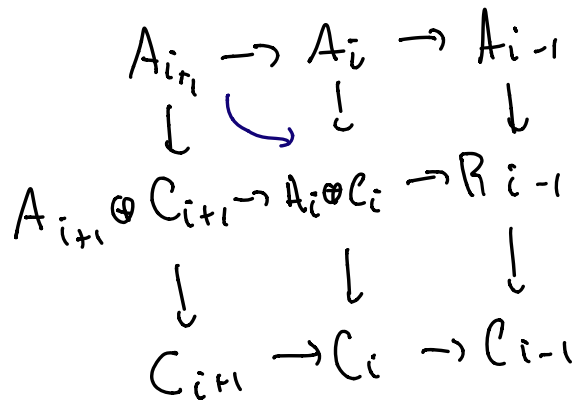


Figure it out.

Some brief foundational comments

All categories are small.

i.e. all cats have sets of objects & morphisms

$\text{Ab}, \text{Set}, \text{Mod}_R$ all subsets of some un-named set ~~"set"~~

Def $\text{HCh}_R^{\geq 0}$ cat. of chain complexes ^{of R -mods}, w/ all negative terms = 0, w/ Morphisms = hom. classes of maps.

Def $\text{HPrCh}_{R, \text{exact} \geq 0}^{\geq 0}$ cat. of ch. complexes of proj. modules, hom classes of morphisms, $\text{deg} \geq 0$ exact except at 0.

$\text{HPrCh}_{\text{exact} \geq 0}^{\geq 0}(A)$

Consider the functor $H_0: \text{HP, Cl}_{\text{exact}}^{\geq 0}(A) \rightarrow A$

Claim: if A has enough projectives, then H_0 is an equiv. of cat.

Pf: enough proj's \Rightarrow essential surjectivity.

$$A \text{ w/ proj's } A_0 \Rightarrow H_0(A_0) = A$$

1st lemma today \Rightarrow fully faithful \square .

From now on — assume that A has enough projectives.

Typical convention choose a quasi-inverse.

Given $F: A \rightarrow B$ right exact, define

$$L_i F: A \rightarrow B \text{ via } L_i F(A) = H_i(F(A_0))$$

given $f: A \rightarrow B$ $L_i F(f)$ induced from $f_0: A_0 \rightarrow B$.

i.e. have f_* (up to hom. eq.)

$$f_* = A_* \rightarrow B_* \quad \text{in } \text{Ch}(A)$$

$$F(f_*): FA_* \rightarrow FB_* \quad \text{in } \text{Ch}(B)$$

$$H_i(F(f_*)): H_i(FA_*) \rightarrow H_i(FB_*)$$

$$\begin{array}{ccc} \text{"} & \text{"} & \text{"} \\ L_i F(f) & L_i F(A) & L_i F(B) \end{array}$$

Note $H_i(F(f_*)) = H_i(F(g_*))$ if $f_* \simeq g_*$
are both extensions of f to $A_* \rightarrow B_*$

Because $F(f_*) \simeq F(g_*)$ are homotopic maps from
 FA_* to FB_*

Because the identity $hd + dh_* = f_* - g_*$
is preserved by any additive functor

Thm $\{L_i F\}: A \rightarrow B$ as above is a δ -functor.

Pf: $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in A

use res. A, i, C , horseshoe to get B'

to get $0 \rightarrow A \rightarrow B' \rightarrow C \rightarrow 0$

apply F , get a SES of complexes in \mathcal{B}

$0 \rightarrow FA \rightarrow FB' \rightarrow FC \rightarrow 0$ (using split
for each i)

get LES

notice: \exists chain maps

$B \rightarrow B'$ unique up to
hom.
comp. w/ $\text{id}_{B \rightarrow B}$

both comps. homotopic to id

therefore, the induced maps $FB \hookrightarrow FB'$

well defined up to hom. both comps

hom to id . induce well defined

maps $H_i(FB) \xrightarrow{\cong} H_i(FB')$

making this identification, get LES

$\rightarrow L_i F(A) \rightarrow L_i F(B) \rightarrow L_i F(C) \rightarrow L_{i-1}(A) \rightarrow \dots$

check: the connecting maps are natural.

Thm $\{L_i F\}$ are universal

Note if P is projective, choose res.

$$P_0 = 0 \rightarrow P \rightarrow 0$$

ind \mathbb{Z}

$$H_i(FP_0) = 0 \text{ if } i > 0$$

$$L_i F(P) = 0 \text{ if } i > 0.$$

enough projectives \Rightarrow effaceable.