<u>Notation</u>: if $A_\bullet$ is a resolution of A then we'll write (today) $A_\bullet^+$ for complex $\to A_2 \to A_1 \to A_0 \to A \to 0$

<u>Lemma</u> Given an abelian cat $\mathcal{A}$, objects A, B, $f: A \to B$, projective resolutions $A_\bullet, B_\bullet$ of A & B then $\exists f_\bullet : A_\bullet \to B_\bullet$ s.t. extends the map $A \to B$ via a map $A_\bullet^+ \to B_\bullet^+$. Further, $f_\bullet$ is unique up to homotopy.

<u>Pf</u>:

exists since A_0 proj, $B_0 \twoheadrightarrow B$.

$$\begin{tikzcd}
A_1 \arrow[r] & A_0 \arrow[r] \arrow[d, dashed] \arrow[dr] & A \arrow[r] \arrow[d, "f"] & 0 \\
B_1 \arrow[r] & B_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

inductively

$$\begin{tikzcd}
A_{i+1} \arrow[r] \arrow[dr, bend left] \arrow[d, dashed] & A_i \arrow[r] \arrow[d] & A_{i-1} \arrow[r] \arrow[d] & {} \\
B_{i+1} \arrow[r, "d_{i+1}"] \arrow[d] & B_i \arrow[r, "d_i"] & B_{i-1} \arrow[r] & {} \\
\ker d_i = \operatorname{im} d_{i+1} \arrow[ur] & & &
\end{tikzcd}$$

etc.

uniqueness up to homotopy:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1, g_1"] \arrow[dl, "h_1"'] & A_0 \arrow[r] \arrow[d, "f_0, g_0"] \arrow[dl, "h_0"'] & A \arrow[r] \arrow[d, "f"] & 0 \\
B_1 \arrow[r] & B_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

want h_i's s.t. $\qquad d f_0 = d g_0 = f d$

$$hd + dh = f - g$$

want $h_0 : A_0 \to B_1$ s.t.

$$d h_0 = f_0 - g_0$$

$d(f_0 - g_0) = 0 \Rightarrow$ by exactness of bottom $f_0 - g_0 = d(?)$

gives

$$\begin{tikzcd} & A_1 \arrow[r] \arrow[d, "{f_1, g_1}"] \arrow[dl, "h_1"'] & A_0 \arrow[r] \arrow[d, "{f_0, g_0}"] \arrow[dl, "h_0"'] \arrow[d, "\text{proj.}"] & A \arrow[r] \arrow[d, "f"] & 0 \\ {} & B_1 \arrow[r] \arrow[d, "\twoheadrightarrow"] & B_0 \arrow[r] & B \arrow[r] & 0 \\ & \operatorname{im} d_1 \arrow[ur, hook] & & & \end{tikzcd}$$

keep going. D.

Remark: seq. on bottom didn't need to be proj. top didn't need to be exact?

<u>Ex</u> confirm or deny ↗

<u>Lemma (Horseshoe)</u>

Given an exact seq $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, proj. res. $A_\bullet \vdots C_\bullet$ of $A \vdots C$, can find proj. res $B_\bullet$ of B s.t. have an exact seq $0 \to A_\bullet^+ \to B_\bullet^+ \to C_\bullet^+ \to 0$ which satisfies the condition that

for each i, $0 \to A_i \to B_i \to C_i \to 0$ is <u>split</u> exact. $i \geq 0$.

"Pf" Set $B_i = A_i \oplus C_i$

define $B_0 \to B$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
A_0 \arrow[r] \arrow[d, "\iota_1"] \arrow[dr, bend left] & A \arrow[d] \\
A_0 \oplus C_0 \arrow[r] \arrow[d, "\pi_2"] & B \arrow[d] \\
C_0 \arrow[r] \arrow[ur, "\mathrm{proj}"'] \arrow[d] & C \arrow[d] \\
0 & 0
\end{tikzcd}$$

add these

general step

$$\begin{tikzcd}
A_{i+1} \arrow[r] \arrow[d] \arrow[dr, bend left] & A_i \arrow[r] \arrow[d] & A_{i-1} \arrow[d] \\
A_{i+1} \oplus C_{i+1} \arrow[r] \arrow[d] & A_i \oplus C_i \arrow[r] \arrow[d] & B_{i-1} \arrow[d] \\
C_{i+1} \arrow[r] & C_i \arrow[r] & C_{i-1}
\end{tikzcd}$$

Figure it out.

Some brief foundational comments

All categories are small.

i.e. all cats have sets of objects & morphisms

Ab, Set, Mod_R all subsets of some un-named set ~~"U"~~

<u>Def</u> $HCh_R^{\geq 0}$ cat. of chain complexes of R-mods w/ all negative terms $= 0$, w/ Morphisms $=$ hom. classes of maps.

<u>Def</u> $HPrCh^{\geq 0}_{R, exact > 0}$ cat of ch. complexes of proj. modules, hom classes of morphisms, deg ≥ 0 exact except at 0.

$HPrCh^{\geq 0}_{exact > 0}(\mathcal{A})$

Consider the functor $H_0 : HProCh^{\geq 0}_{exact > 0}(\mathcal{A}) \to \mathcal{A}$

<u>Claim</u>: if $\mathcal{A}$ has enough projectives, then H_0 is an equiv. of cat.

<u>Pf</u>: enough proj's $\Rightarrow$ essential surjectivity.

A w/ proj res $A_\bullet$ $\Rightarrow$ $H_0(A_\bullet) = A$

1st Lemma today $\Rightarrow$ fully faithful $\square$.

From now on — assume that $\mathcal{A}$ has enough projectives.

<u>Typical convention</u> choose a quasi-inverse.

Given $F : \mathcal{A} \to \mathcal{B}$ right exact, define

$L_iF : \mathcal{A} \to \mathcal{B}$ via $L_iF(A) = H_i(F(A_\bullet))$

given $f : A \to B$ $L_iF(f)$ induced from $f_\bullet : A_\bullet \to B_\bullet$

i.e. have $f_\circ$ (up to hom. eq.)

$$f_\circ : A_\circ \to B_\circ \quad \text{in } Ch(\mathcal{A})$$

$$F(f_\circ) : FA_\circ \to FB_\circ \quad \text{in } Ch(\mathcal{B})$$

$$\begin{array}{ccc} H_i(F(f_\circ)) : & H_i(FA_\circ) \to & H_i(FB_\circ) \\ \parallel & \parallel & \parallel \\ L_iF(f) & L_iF(A) & L_iF(B) \end{array}$$

Note $H_i(F(f_\circ)) = H_i(F(g_\circ))$ if $f_\circ$ & $g_\circ$ are both extensions of f to $A_\circ \to B_\circ$

Because $F(f_\circ)$ & $F(g_\circ)$ are homotopic maps from $FA_\circ$ to $FB_\circ$

Because the identity $h_\circ d + d h_\circ = f_\circ - g_\circ$ is preserved by any additive functor

<u>Thm</u> $\{L_iF\} : \mathcal{A} \to \mathcal{B}$ as above is a δ-functor.

<u>Pf:</u> $0 \to A \to B \to C \to 0$ in $\mathcal{A}$

use res. $A_\bullet$ & $C_\bullet$, horshoe to get $B_\bullet'$

to get $0 \to A_\bullet \to B_\bullet' \to C_\bullet \to 0$

apply F, get a SES of complexes in $\mathcal{B}$

$0 \to FA_\bullet \to FB_\bullet' \to FC_\bullet \to 0$ (using split for each i)

get LES

notice: $\exists$ chain maps

$B_\bullet \rightleftarrows B_\bullet'$ unique up to hom.

comp. of id $B \to B$

both comps. homotopic to id

therefore, the induced maps $FB_\bullet \rightleftarrows FB_\bullet'$ well defined up to hom. both comps hom to id. induce well defined isos $H_i(FB_\bullet) \rightleftarrows H_i(FB_\bullet')$

making this identification, get LES

$\to L_iF(A) \to L_iF(B) \to L_iF(C) \to L_{i-1}(A) \to \ldots$

Check: the connecting maps are natural.

Thm $\{L_i F\}$ are universal

Note if P is projective, choose res.

$P_\bullet = 0 \to P \to 0$
in deg 0

$H_i(FP_\bullet) = 0$ if $i > 0$

$L_i F(P) = 0 \quad i > 0.$

enough projectives $\Rightarrow$ effaceable.