Ab-Cat-Recap

$$\mathcal{A} \longrightarrow \underset{\text{additive!}}{\overset{F}{\mathrm{Fun}(\mathcal{A}^{op}, Ab)}} \xrightarrow[\text{Fix } Y \quad F \mapsto F(Y)]{e_Y} Ab$$

$$A \longmapsto \underbrace{(X \mapsto \mathrm{Hom}(X,A))}_{h_A}$$

Composition: $\mathcal{A} \longrightarrow Ab$ $\quad \mathrm{Hom}(Y,-)$ left, but not right exact

$$A \longmapsto \mathrm{Hom}(Y,A)$$

Exercise show that e_Y exact & a map is monic/epic in $\mathrm{Fun}(\mathcal{A}^{op}, Ab) \Longleftrightarrow e_Y(\text{map})$ is monic/epic for all Y.

Define $\mathcal{E}$ = full subcat of $\mathrm{Fun}(\mathcal{A}^{op}, Ab)$ whose objects are $\{F \mid \forall X \in \mathcal{A}, \forall a \in FX, \exists Y \overset{f}{\twoheadrightarrow} X \text{ s.t. } (Ff)(a) = 0\}$

"weakly coeffacable functors"

Fact: $\mathcal{E}$ is a Serre subcategory &

$\mathcal{A} \longrightarrow Fun(\mathcal{A}^{op}, Ab)/\mathcal{C}$ <u>is exact</u>.

<u>Moreover:</u>

If we let $\mathcal{L} \subset Fun(\mathcal{A}^{op}, Ab)$ be the full subcategory of left exact functors then

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[rr, bend right, "\sim", "\text{equivalence}"'] & Fun(\mathcal{A}^{op}, Ab) \arrow[r] & Fun(\mathcal{A}^{op}, Ab)/\mathcal{C} \end{tikzcd}$$

<u>Why?</u>

Quick outline of localization / quotient of Abelian cats.

$\mathcal{A}$ Abelian category $\mathcal{C}$ a full subcategory

$\mathcal{C}$ is called a Serre subcat (aka. thick, épaisse) if $\forall\ 0 \to A' \to A \to A'' \to 0$ in $SES_{\mathcal{A}}$,

$A', A'' \in \mathcal{C} \iff A \in \mathcal{C}$.

<u>Thm</u> (Serre) $\exists$ An Abelian cat $\mathcal{A}/\mathcal{C}$ & an exact functor $\mathcal{A} \xrightarrow{T} \mathcal{A}/\mathcal{C}$ s.t.

$\forall a \in \mathcal{C}$, $Ta \simeq 0$ & if $\mathcal{A} \xrightarrow{T'} \mathcal{B}$ is any other exact functor w/ $T'a \cong 0$ for all $a \in \mathcal{C}$,

then $\exists!$ $F: \mathcal{A}/\mathcal{C} \to \mathcal{B}$ s.t.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "T"] \arrow[dr, "T'"'] & \mathcal{A}/\mathcal{C} \arrow[d, "F"] \\ & \mathcal{B} \end{tikzcd}$$

Construction of $\mathcal{A}/\mathcal{C}$:

$\mathrm{ob}(\mathcal{A}/\mathcal{C}) = \mathrm{ob}(\mathcal{A})$

given $A' \xrightarrow{f} A$, $B' \xrightarrow{g} B$ in $\mathcal{A}$, have a natural map

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}}(A', \mathrm{coker}\, g)$$

$$\begin{tikzcd} A' \arrow[dr] & & & \mathrm{coker}\, g \\ & A \arrow[r] & B \arrow[ur] & \end{tikzcd}$$

let $\Lambda_{A,B} = \{ (A' \to A, B' \to B)$ monic s.t. $A/A', B' \in \mathcal{C} \}$

this is a directed set and we define

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(A,B) = \varinjlim_{(A',B') \in \Lambda_{A,B}} \mathrm{Hom}_{\mathcal{A}}(A', B/\mathrm{im}\, B')$$

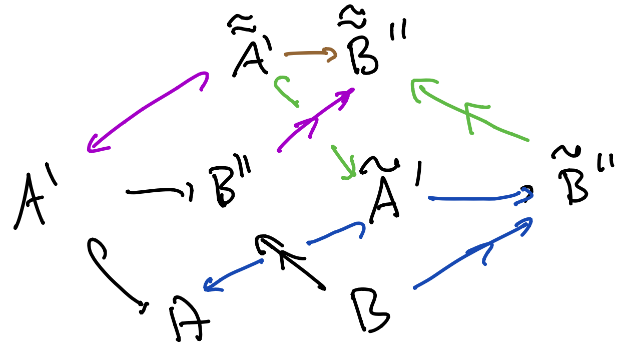
<u>Notation</u>: a map $A \to B$ in $\mathcal{A}$ is $\mathcal{C}$-epic if coker $\in \mathcal{C}$

is $\mathcal{C}$ monic if ker $\in \mathcal{C}$

& is $\mathcal{C}$-iso if both the above.

homs $A \to B$ in $\mathcal{A}/\mathcal{C}$ are equiv classes of

homs
$$\begin{array}{ccc} A' & \longrightarrow & B'' \\ {\scriptstyle \mathcal{C}\text{ epic}}\downarrow & & \uparrow{\scriptstyle \mathcal{C}\text{-monic}} \\ A & & B \end{array}$$



<u>Example</u> R comm. ring, $S \subset R$ a multiplicative set.

$$\mathcal{C} = \langle \{ M \in \mathrm{Mod}_R \mid \forall m \in M,\ sm = 0 \text{ some } s \in S \} \rangle$$

Then $\mathcal{C}$ is thick & $\mathrm{Mod}_R / \mathcal{C} \simeq \mathrm{Mod}_{RS^{-1}}$

In this case the quotient cat is a "localization"

In fact, there is a right adjoint

$$\mathrm{Mod}_{RS^{-1}} \longrightarrow \mathrm{Mod}_R \quad \text{and}$$

$$\begin{tikzcd}
\mathrm{Mod}_{RS^{-1}} \arrow[r, "\iota"] \arrow[rr, bend right, "\sim \mathrm{id}"'] & \mathrm{Mod}_R \arrow[r, "\text{“loc”}"] & \mathrm{Mod}_{RS^{-1}}
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{Mod}_R \arrow[r, "\mathrm{loc}"] \arrow[rr, bend right, "\text{“localization”}"'] & \mathrm{Mod}_{RS^{-1}} \arrow[r, "?"] & \mathrm{Mod}_K
\end{tikzcd}
$$

<u>Prop</u> if $T : A \to A/\mathcal{C}$ has a right adjoint S then $TS \simeq \mathrm{id}_{A/\mathcal{C}}$ and so $\underbrace{S(A/\mathcal{C})}_{\mathcal{B}} \simeq A/\mathcal{C}$

Language: we say in this case that T (or $\mathcal{B}$ or $\mathcal{C}$) is localizing.

Back to homological algebra

"Last time" we constructed left derived functors for a right exact functor $F : \mathcal{A} \to \mathcal{B}$ in the case that $\mathcal{A}$ had enough projectives.

i.e. a δ functor, universal $\{F_i\}$, $F_0 = F$

Reversing arrows, we find if $F : \mathcal{A} \to \mathcal{B}$ is left exact, and if $\mathcal{A}$ has enough injectives then $\exists$ univ. δ-functor $\{F^i\}$ $F^0 = F$

"right derived functors of F"

$R^iF = F^i$

We showed that Mod_R has enough projectives.

In fact Mod_R also has enough injectives.