

Math 1610, Honors Calculus 1, Fall 2026, Exam 1 — Solutions

Please note. These solutions were generated agentically (by an AI assistant). They have been looked over, but they may still contain errors. If you find a mistake, or anything that is unclear or could be better explained, please let us know — we would be glad to hear about it.

These are written out slowly, with the reasoning spelled out at each step. In most cases a correct solution on the exam could be much shorter than what appears here.

1. Let $f : X \rightarrow Y$ be a function and $C, D \subseteq Y$. Show that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Solution: Everything here comes from unwinding two definitions, so let us write them down first.

Saying $x \in f^{-1}(C)$ means exactly: $f(x) \in C$. That is the definition we were given.

Saying $z \in C \cap D$ means exactly: $z \in C$ and $z \in D$.

To prove two sets are equal, we show that each one is contained in the other. So there are two things to do.

First: $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

Let x be any element of $f^{-1}(C \cap D)$. We must show that x belongs to $f^{-1}(C) \cap f^{-1}(D)$.

Since $x \in f^{-1}(C \cap D)$, the definition of preimage says

$$f(x) \in C \cap D.$$

Since $f(x)$ is in the intersection, it is in both sets:

$$f(x) \in C \quad \text{and} \quad f(x) \in D.$$

Now read each of these backwards through the definition of preimage. From $f(x) \in C$ we get $x \in f^{-1}(C)$. From $f(x) \in D$ we get $x \in f^{-1}(D)$. So x lies in both of those sets, which is exactly what it means to say

$$x \in f^{-1}(C) \cap f^{-1}(D).$$

Second: $f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$.

Let x be any element of $f^{-1}(C) \cap f^{-1}(D)$. Then x is in both sets, so

$$x \in f^{-1}(C) \quad \text{and} \quad x \in f^{-1}(D).$$

Reading the definition forwards this time, the first gives $f(x) \in C$ and the second gives $f(x) \in D$. So $f(x)$ lies in both C and D , i.e.

$$f(x) \in C \cap D,$$

and therefore $x \in f^{-1}(C \cap D)$.

Each set is contained in the other, so the two sets are equal.

2. $A' \subseteq A$ and $B' \subseteq B$ are nonempty subsets of $\mathbb{R}$, and $a < b$ for every $a \in A$ and every $b \in B$.

(a) Show that $\sup A \leq \inf B$. Can they be equal?

Solution: First, let us make sure the two numbers in the statement actually exist.

Pick any single element $b_0 \in B$ (we can, since B is nonempty). Every $a \in A$ satisfies $a < b_0$, so A is bounded above. Since A is also nonempty, $\sup A$ exists. The same argument with the roles reversed shows B is bounded below by any element of A , so $\inf B$ exists.

Now the inequality. The argument has two halves, and each half is the same move: show something is a bound, then use that a sup is the least upper bound and an inf is the greatest lower bound.

Step 1: every element of B is an upper bound for A .

Let $b \in B$. By hypothesis $a < b$ for every $a \in A$, and in particular $a \leq b$ for every $a \in A$. That is precisely what it means for b to be an upper bound for A .

Step 2: therefore $\sup A \leq b$ for every $b \in B$.

We just showed each $b \in B$ is an upper bound for A . And $\sup A$ is the smallest upper bound for A . A smallest thing is no bigger than any of the things it is smallest among, so $\sup A \leq b$.

Step 3: therefore $\sup A$ is a lower bound for B .

Step 2 says $\sup A \leq b$ for every single $b \in B$. That is exactly what it means for $\sup A$ to be a lower bound for B .

Step 4: therefore $\sup A \leq \inf B$.

$\inf B$ is the largest lower bound for B , and by Step 3 our number $\sup A$ is one of the lower bounds. A largest thing is no smaller than any of the things it is largest among, so $\sup A \leq \inf B$.

Can they be equal? Yes. Take

$$A = (0, 1), \quad B = (1, 2).$$

Every $a \in A$ satisfies $a < 1$ and every $b \in B$ satisfies $b > 1$, so indeed $a < b$ for every such pair, as the hypothesis requires. But

$$\sup A = 1 \quad \text{and} \quad \inf B = 1,$$

so the two are equal. (Notice that neither value is actually attained: $1 \notin A$ and $1 \notin B$. That is what lets the strict inequality $a < b$ hold for every pair while the sup and inf still meet.)

- (b) Show that $\sup A' \leq \sup A$ and $\inf B' \geq \inf B$.

Solution: This is the statement that making a set smaller cannot make its supremum bigger, and cannot make its infimum smaller.

The supremum. We want $\sup A' \leq \sup A$.

Take any $a \in A'$. Since $A' \subseteq A$, this a is also an element of A . And $\sup A$ is an upper bound for A , so $a \leq \sup A$.

That was true for every $a \in A'$, so the number $\sup A$ is an upper bound for A' .

Now $\sup A'$ is the smallest upper bound for A' , and we have just exhibited $\sup A$ as one of the

upper bounds for A' . So

$$\sup A' \leq \sup A.$$

The infimum. We want $\inf B' \geq \inf B$. The argument is the same with every inequality turned around.

Take any $b \in B'$. Since $B' \subseteq B$, this b is in B , and $\inf B$ is a lower bound for B , so $b \geq \inf B$. That was true for every $b \in B'$, so $\inf B$ is a lower bound for B' . Since $\inf B'$ is the largest lower bound for B' ,

$$\inf B' \geq \inf B.$$

(c) Show that if $\sup A' = \inf B'$ then $\sup A = \inf B$.

Solution: The idea is to line up all four numbers in a single chain of inequalities, and then notice that the two ends are equal.

Here is the chain:

$$\sup A' \leq \sup A \leq \inf B \leq \inf B'.$$

Each link was proved above:

- $\sup A' \leq \sup A$ is the first half of part (b).
- $\sup A \leq \inf B$ is part (a).
- $\inf B \leq \inf B'$ is the second half of part (b), written the other way round.

Now suppose $\sup A' = \inf B'$. Then the leftmost and rightmost numbers in the chain are the same number. But if $p \leq q \leq r \leq s$ and $p = s$, then there is no room anywhere: every one of those inequalities must actually be an equality, since any strict inequality would force $p < s$.

In particular the middle link is an equality, which says

$$\sup A = \inf B.$$

3. Use the definition of a limit to show that $\lim_{x \rightarrow 4} \frac{1}{x} = \frac{1}{4}$.

Solution: What we must produce: given any $\epsilon > 0$, a number $\delta > 0$ such that

$$0 < |x - 4| < \delta \quad \implies \quad \left| \frac{1}{x} - \frac{1}{4} \right| < \epsilon.$$

Start by rewriting the thing we want to be small. Putting the two fractions over a common denominator,

$$\left| \frac{1}{x} - \frac{1}{4} \right| = \left| \frac{4 - x}{4x} \right| = \frac{|x - 4|}{4|x|}.$$

The numerator $|x - 4|$ is exactly the quantity we control — it is what δ bounds. The denominator is the problem: if x is allowed near 0, then $4|x|$ is tiny and the whole fraction blows up. So we deal with the denominator first, and only then worry about ϵ .

Step 1: keep x away from 0. Suppose we insist that $|x - 4| < 2$. That says x is within 2 of 4, so

$$2 < x < 6.$$

In particular $|x| > 2$, so $4|x| > 8$, and therefore

$$\frac{1}{4|x|} < \frac{1}{8}.$$

Combining this with the rewriting above, as long as $|x - 4| < 2$, we have

$$\left| \frac{1}{x} - \frac{1}{4} \right| = \frac{|x - 4|}{4|x|} < \frac{|x - 4|}{8}.$$

Step 2: now make that small. We want $\frac{|x-4|}{8} < \epsilon$, which happens exactly when $|x - 4| < 8\epsilon$.

Step 3: ask for both at once. Let

$$\delta = \min\{2, 8\epsilon\}.$$

This is positive, since both 2 and 8ϵ are.

Suppose $0 < |x - 4| < \delta$. Then:

- $|x - 4| < 2$, so Step 1 applies and $\left| \frac{1}{x} - \frac{1}{4} \right| < \frac{|x-4|}{8}$;
- $|x - 4| < 8\epsilon$, so $\frac{|x-4|}{8} < \epsilon$.

Putting these together,

$$\left| \frac{1}{x} - \frac{1}{4} \right| < \frac{|x - 4|}{8} < \frac{8\epsilon}{8} = \epsilon,$$

which is what we wanted.

(The number 2 in Step 1 is not special — any choice keeping x a fixed distance from 0 works, with a different constant in Step 2.)

4. F is an ordered field and $a, b \in F$ are nonzero with $a < b$ and $1/a < 1/b$. Show that ab is negative.

Solution: Since a and b are both nonzero, their product ab is nonzero too. So by trichotomy, ab is either positive or negative, and we want to rule out “positive”. We argue by contradiction.

Suppose $ab > 0$.

First, $1/(ab)$ is also positive. Suppose it were negative. A positive number times a negative number is negative, so $ab \cdot \frac{1}{ab}$ would be negative. But that product is 1, and $1 = 1^2$ is positive. So $1/(ab)$ cannot be negative, and it is not zero, so it is positive.

Now multiply the inequality $a < b$ by this positive number. Multiplying an inequality by a positive element preserves it, so

$$a \cdot \frac{1}{ab} < b \cdot \frac{1}{ab}.$$

Simplify both sides. On the left, the a 's cancel:

$$a \cdot \frac{1}{ab} = \frac{a}{ab} = \frac{1}{b}.$$

On the right, the b 's cancel:

$$b \cdot \frac{1}{ab} = \frac{b}{ab} = \frac{1}{a}.$$

So the inequality reads

$$\frac{1}{b} < \frac{1}{a}.$$

But this contradicts what we were given. We were told $\frac{1}{a} < \frac{1}{b}$, and we have just derived $\frac{1}{b} < \frac{1}{a}$. Both cannot hold.

So the assumption $ab > 0$ is impossible, and therefore ab is negative.

(What this says in familiar terms: a and b must have opposite signs. Taking reciprocals reverses the order only when the two numbers sit on opposite sides of 0.)

5. For bounded $f, g : [a, b] \rightarrow \mathbb{R}$ with $f(x) \leq g(x)$ for all x , show that $L \int_a^b f(x) dx \leq L \int_a^b g(x) dx$.

Solution: Both sides are suprema of sets of numbers, so let us name those sets:

$$S_f = \left\{ \int_a^b s(x) dx \mid s \text{ is a step function with } s \leq f \right\},$$

and S_g the same thing with g in place of f . By definition

$$L \int_a^b f = \sup(S_f) \quad \text{and} \quad L \int_a^b g = \sup(S_g).$$

The key observation: $S_f \subseteq S_g$.

Why? Take any number in S_f . By the way S_f was defined, that number is $\int_a^b s$ for some step function s satisfying $s(x) \leq f(x)$ for every x .

But we are told $f(x) \leq g(x)$ for every x . Chaining the two inequalities, for every x ,

$$s(x) \leq f(x) \leq g(x), \quad \text{so} \quad s(x) \leq g(x).$$

So this same step function s is also one of the step functions sitting below g — which means $\int_a^b s$ is one of the numbers in S_g .

That is: every number in S_f is also in S_g , i.e. $S_f \subseteq S_g$.

Finishing. Now we only need: a supremum over a smaller set is no bigger than a supremum over a larger one.

Here is why. Every element of S_f is an element of S_g , and $\sup(S_g)$ is an upper bound for S_g . So every element of S_f is $\leq \sup(S_g)$, which makes $\sup(S_g)$ an upper bound for S_f as well. Since $\sup(S_f)$ is the smallest upper bound for S_f ,

$$\sup(S_f) \leq \sup(S_g).$$

Translating back,

$$L \int_a^b f(x) dx \leq L \int_a^b g(x) dx.$$

(Notice that we never compared a particular step function below f with a particular step function below g . The whole argument is about the two collections of competitors, one sitting inside the other.)