Math 1610, Honors Calculus 1, Fall 2026, Homework 2 — Solutions

Please note. These solutions were generated agentically (by an AI assistant). They have been looked over, but they may still contain errors. If you find a mistake, or anything that is unclear or could be better explained, please let us know — we would be glad to hear about it.

Throughout, F is an ordered field with set of positive elements F^+, and $x < y$ means $y - x \in F^+$. We use freely that F^+ is closed under addition and multiplication, that exactly one of $x \in F^+$, $-x \in F^+$, $x = 0$ holds, and the field identity $(-x)y = -(xy)$.

1. Show that if $a < 0$ and $b < c$ then $ac < ab$.

Solution: Since $a < 0$ we have $0 - a = -a \in F^+$, and since $b < c$ we have $c - b \in F^+$. As F^+ is closed under multiplication,

$$(-a)(c - b) \in F^+.$$

Expanding, $(-a)(c - b) = -(a(c - b)) = -(ac - ab) = ab - ac$. So $ab - ac \in F^+$, which is precisely the statement $ac < ab$.

2. Show that if $a \in F$ then $a^2 \geq 0$, and that $a^2 = 0$ if and only if $a = 0$.

Solution: By the order axioms, exactly one of the following holds: $a = 0$, or $a \in F^+$, or $-a \in F^+$. We take the three cases in turn.

Case $a = 0$. Then $a^2 = 0 \cdot 0 = 0$, so $a^2 \geq 0$.

Case $a \in F^+$. Then $a^2 = a \cdot a \in F^+$ since F^+ is closed under multiplication, so $a^2 > 0$.

Case $-a \in F^+$. Then $(-a)(-a) \in F^+$. Applying $(-x)y = -(xy)$ twice,

$$(-a)(-a) = -\big(a(-a)\big) = -\big(-(aa)\big) = a^2,$$

so again $a^2 \in F^+$, i.e. $a^2 > 0$.

In every case $a^2 \geq 0$, which proves the first claim.

For the second: if $a = 0$ then $a^2 = 0$ by the first case. Conversely, if $a \neq 0$ then one of the last two cases applies and gives $a^2 \in F^+$; since $0 \notin F^+$ this forces $a^2 \neq 0$. Taking the contrapositive, $a^2 = 0$ implies $a = 0$.

3. Show that if $a > b > 0$ and $c > d > 0$ then $ac > bd$.

Solution: Since $a > b$ we have $a - b \in F^+$, and since $c > d > 0$ we have $c \in F^+$. Hence

$$(a - b)c = ac - bc \in F^+, \qquad \text{that is,} \qquad ac > bc.$$

Since $c > d$ we have $c - d \in F^+$, and since $b > 0$ we have $b \in F^+$. Hence

$$b(c-d) = bc - bd \in F^+, \qquad \text{that is,} \qquad bc > bd.$$

Adding the two positive elements $ac - bc$ and $bc - bd$, which is again in F^+, gives

$$(ac - bc) + (bc - bd) = ac - bd \in F^+,$$

so $ac > bd$.

4. Lower bounds and greatest lower bounds.

(a) Show that if $b, d \in \mathbb{R}$ are both greatest lower bounds for A, then $b = d$.

Solution: Since d is in particular a lower bound for A, and b is a greatest lower bound, the defining property of b gives $d \leq b$.

Exchanging the roles of b and d: since b is a lower bound for A and d is a greatest lower bound, we get $b \leq d$.

Now suppose $b \neq d$. Then $d \leq b$ forces $d < b$, and $b \leq d$ forces $b < d$, so both $b - d \in F^+$ and $d - b = -(b-d) \in F^+$. This is impossible, since the order axioms never allow both x and $-x$ to be positive. Hence $b = d$.

(b) Show that if $A \subseteq \mathbb{R}$ is a nonempty set which is bounded below, then it has an infimum, and $\inf(A) = -\sup(-A)$, where $-A = \{-a \mid a \in A\}$.

Solution: First we check that $-A$ is nonempty and bounded above, so that its supremum exists.

Since $A \neq \varnothing$ there is some $a \in A$, and then $-a \in -A$, so $-A \neq \varnothing$. Since A is bounded below there is some b with $b \leq a$ for all $a \in A$; negating reverses the inequality, so $-a \leq -b$ for all $a \in A$, i.e. $-b$ is an upper bound for $-A$.

By completeness of $\mathbb{R}$, $s = \sup(-A)$ exists. We claim $-s$ is a greatest lower bound for A.

$-s$ is a lower bound for A. Let $a \in A$. Then $-a \in -A$, so $-a \leq s$ because s is an upper bound for $-A$. Negating gives $-s \leq a$.

$-s$ is the greatest one. Let c be any lower bound for A, so $c \leq a$ for every $a \in A$. Negating, $-a \leq -c$ for every $a \in A$, so $-c$ is an upper bound for $-A$. Since s is the least upper bound, $s \leq -c$, and negating again gives $c \leq -s$.

So A has an infimum and, by part (a) it is unique, so

$$\inf(A) = -s = -\sup(-A).$$

5. Consider $S_1 = \{x \in \mathbb{Q} \mid x^2 \geq 2x + 1\}$ and $S_2 = \{x \in \mathbb{Q} \mid x^2 \leq 2x + 1\}$.

Solution: It is worth describing the two sets first. Completing the square,

$$x^2 - 2x - 1 = (x-1)^2 - 2,$$

so $x^2 - 2x - 1 \le 0$ exactly when $(x-1)^2 \le 2$, i.e. exactly when $1 - \sqrt{2} \le x \le 1 + \sqrt{2}$. Hence

$$S_2 = \mathbb{Q} \cap [\,1 - \sqrt{2},\ 1 + \sqrt{2}\,], \qquad S_1 = \mathbb{Q} \cap \big((-\infty, 1 - \sqrt{2}\,] \cup [\,1 + \sqrt{2}, \infty)\big).$$

Note that $1 \pm \sqrt{2}$ are irrational (if $1 + \sqrt{2}$ were rational then so would be $\sqrt{2}$), so the endpoints themselves belong to neither set.

(a) Do S_1 or S_2 have suprema within $\mathbb{Q}$?

Solution: Neither does, but for different reasons.

$\underline{S_1.}$ S_1 contains every rational number greater than $1 + \sqrt{2}$, and there is no rational number larger than all of these. So S_1 has no upper bound at all in $\mathbb{Q}$, and hence no supremum.

$\underline{S_2.}$ Here S_2 is nonempty ($0 \in S_2$, since $0 \le 1$) and bounded above (for instance by 3, since $1 + \sqrt{2} < 3$). But it has no <u>least</u> upper bound in $\mathbb{Q}$.
Suppose $r \in \mathbb{Q}$ were an upper bound for S_2. If $r < 1 + \sqrt{2}$, then by density of the rationals we could choose a rational q with $\max(r, 1 - \sqrt{2}) < q < 1 + \sqrt{2}$; such a q lies in S_2 and satisfies $q > r$, contradicting that r is an upper bound. So $r \ge 1 + \sqrt{2}$, and since r is rational and $1 + \sqrt{2}$ is not, in fact $r > 1 + \sqrt{2}$.
But then, again by density, there is a rational q with $1 + \sqrt{2} < q < r$. This q is also an upper bound for S_2, and $q < r$. So no rational upper bound is least, and S_2 has no supremum in $\mathbb{Q}$.

(b) Do S_1 or S_2 have suprema within $\mathbb{R}$?

Solution: $\underline{S_1}$ still has no supremum: it is unbounded above in $\mathbb{R}$ for the same reason as before, and an unbounded set has no upper bound to be least among.

$\underline{S_2}$ does. It is nonempty and bounded above, so by the completeness axiom its supremum exists in $\mathbb{R}$, and we claim it equals $1 + \sqrt{2}$.
It is an upper bound: every $x \in S_2$ satisfies $x \le 1 + \sqrt{2}$ by the description above.
It is least: suppose $u < 1 + \sqrt{2}$. By density of $\mathbb{Q}$ choose a rational q with $\max(u, 1 - \sqrt{2}) < q < 1 + \sqrt{2}$. Then $q \in S_2$ and $q > u$, so u is not an upper bound for S_2.
Hence $\sup(S_2) = 1 + \sqrt{2}$.

This is exactly the phenomenon that motivates the completeness axiom: S_2 is a perfectly reasonable bounded set of rational numbers whose least upper bound is missing from $\mathbb{Q}$ and present in $\mathbb{R}$.