Math 1610, Honors Calculus 1, Fall 2026, Homework 2

1. Suppose that F is an ordered field. Show that if $a < 0$ and $b < c$ then $ac < ab$.

2. Suppose that F is an ordered field. Show that if $a \in F$ then $a^2 \geq 0$ and $a^2 = 0$ if and only if $a = 0$.

3. Suppose that F is an ordered field. Show that if $a > b > 0$ and $c > d > 0$ then $ac > bd$.

4. Let $A \subseteq R$ be a subset. We say that $b \in \mathbb{R}$ is a lower bound for A if $b \leq a$ for every $a \in A$. We say that b is a greatest lower bound (or infemum) if b is a lower bound and for every other lower bound c for A we have $c \leq b$.

   (a) Show that if $b, d \in \mathbb{R}$ are both greatest lower bounds for A, then $b = d$.

   (b) By the prior part, it makes sense to write $inf(A)$ to denote the infemum of A (when it exists) and $sup(B)$ to denote the supremum of a set $B \subseteq R$ (when it exists).

   Show that if $A \subseteq \mathbb{R}$ is a nonempty set which is bounded below, it has an infemum, and $inf(A) = -sup(-A)$ where we define $-A = \{-a \mid a \in A\}$.

5. Consider the sets $S_1 = \{x \in \mathbb{Q} \mid x^2 \geq 2x + 1\}$ and $S_2 = \{x \in \mathbb{Q} \mid x^2 \leq 2x + 1\}$ as a subset of $\mathbb{Q}$.

(a) Do S_1 or S_2 have suprema within $\mathbb{Q}$? Justify your answer.

(b) Do S_1 or S_2 have suprema within $\mathbb{R}$? Justify your answer.