

Math 1610, Honors Calculus 1, Fall 2026, Homework 3 — Solutions

Please note. These solutions were generated agentially (by an AI assistant). They have been looked over, but they may still contain errors. If you find a mistake, or anything that is unclear or could be better explained, please let us know — we would be glad to hear about it.

We use the density theorem proved in class: if $a < b$ are real numbers, then there is a rational number r with $a < r \leq b$.

1. (a) Show that if a is rational and b is not rational, then $a + b$ is not rational, and that ab is not rational if $a \neq 0$.

Solution: Both statements are proved by contradiction, using that $\mathbb{Q}$ is closed under subtraction and under division by nonzero elements.

Sum. Suppose $a + b$ were rational, say $a + b = q$ with $q \in \mathbb{Q}$. Then

$$b = q - a,$$

and $q, a \in \mathbb{Q}$, so $q - a \in \mathbb{Q}$. This says b is rational, contrary to hypothesis. Hence $a + b$ is irrational.

Product. Suppose $a \neq 0$ and ab were rational, say $ab = q$ with $q \in \mathbb{Q}$. Since $a \neq 0$ we may divide:

$$b = \frac{q}{a},$$

and $q, a \in \mathbb{Q}$ with $a \neq 0$, so $q/a \in \mathbb{Q}$. Again this makes b rational, a contradiction. Hence ab is irrational.

(The hypothesis $a \neq 0$ is needed: if $a = 0$ then $ab = 0$ is rational for every b .)

- (b) Suppose $a < b$ are real numbers. Show that there is an irrational number t with $a < t \leq b$.

Solution: Since $a < b$, subtracting $\sqrt{2}$ from both sides gives

$$a - \sqrt{2} < b - \sqrt{2}.$$

These are two real numbers with the first less than the second, so the density theorem provides a rational number r with

$$a - \sqrt{2} < r \leq b - \sqrt{2}.$$

Set $t = r + \sqrt{2}$. Adding $\sqrt{2}$ throughout the display above gives

$$a < t \leq b,$$

so t lies in the required range.

Finally, t is irrational: $\sqrt{2}$ is irrational and r is rational, so by part (a) the sum $r + \sqrt{2}$ is irrational.

2. Let $S = \{x \in \mathbb{Q} \mid x \leq \pi\} \subseteq \mathbb{R}$. Show that $\sup(S) = \pi$.

Solution: First note S is nonempty, since for instance $0 \in \mathbb{Q}$ and $0 \leq \pi$, so $0 \in S$.

π is an upper bound for S . This is immediate from the definition of S : every element $x \in S$ satisfies $x \leq \pi$.

π is the least upper bound. Let u be any real number with $u < \pi$; we show u is not an upper bound for S . Applying the density theorem to the pair $u < \pi$, there is a rational number r with

$$u < r \leq \pi.$$

Since r is rational and $r \leq \pi$, we have $r \in S$. Since $r > u$, the number u is not an upper bound for S .

So π is an upper bound, and no number smaller than π is an upper bound. Therefore π is the least upper bound, i.e. $\sup(S) = \pi$.

(Notice that the supremum is not attained: $\pi \notin S$, because π is not rational. The argument never needed to know anything about π beyond its being a real number — the same proof shows $\sup\{x \in \mathbb{Q} \mid x \leq c\} = c$ for every real c .)