

Math 1610, Honors Calculus 1, Fall 2026, Homework 6

1. Let $I = (a, b)$ be an open interval, possibly unbounded, and suppose f is differentiable on I with $|f'(x)| \leq M$ for every $x \in I$.
 - (a) Show that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I$.

(b) Show that f is uniformly continuous on I .

(Note that the theorem from lecture — a continuous function on a closed bounded interval is uniformly continuous — doesn't apply here.)

2. In this problem you may use that $(\sin x)' = \cos x$, together with the basic properties of $\sin$ and $\cos$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

- (a) Show that f is differentiable at every $x \neq 0$, and that for such x

$$f'(x) = 2x \sin(1/x) - \cos(1/x).$$

(Product rule and chain rule. Recall $(1/x)' = -1/x^2$.)

(b) Show that f is also differentiable at 0, and that $f'(0) = 0$.

(Here the rules are no help, since f is given by a different formula at 0 — go back to the definition. The difference quotient is $h \sin(1/h)$, and $|\sin t| \leq 1$ for every t .)

(c) Show that $f'(\frac{1}{2\pi n}) = -1$ for every positive integer n , and deduce that f' is not continuous at 0.

So a function can be differentiable everywhere and still have a derivative that is not continuous. Keep this example in mind for the next problem.

3. Let f be differentiable on an open interval containing $[a, b]$, so that in particular $f'(a)$ and $f'(b)$ make sense. Keep in mind that this does not say that f' is continuous.
- (a) Suppose $f'(a) < 0 < f'(b)$. Show that f does not attain its minimum value on $[a, b]$ at either endpoint, and deduce that $f'(c) = 0$ for some $c \in (a, b)$.

(b) Deduce that if $f'(a) < \lambda < f'(b)$, then $f'(c) = \lambda$ for some $c \in (a, b)$.

(So derivatives have the intermediate value property, whether or not they are continuous. This is Darboux's theorem.)

- (c) Explain why part (b) does not simply follow from applying the intermediate value theorem to the function f' .

4. Throughout this problem g is a continuous function on $\mathbb{R}$.

(a) Let $G(x) = \int_0^{x^2} g(t) dt$. Show that G is differentiable and that $G'(x) = 2x g(x^2)$.

(b) Let u and v be differentiable functions on $\mathbb{R}$ and set

$$H(x) = \int_{u(x)}^{v(x)} g(t) dt.$$

Show that H is differentiable and that

$$H'(x) = g(v(x)) v'(x) - g(u(x)) u'(x).$$