Honors Calculus 1610

Lecture 1: Mathematical Context, Sets, and Functions

August 28, 2026

What is this course about?

Our main goal

To understand calculus in the context of modern, rigorous mathematical thought: not only how to use calculus, but how it is built and how we know that it works.

- We will use multivariable calculus as a setting in which to practice.
- We will read, evaluate, and construct mathematical arguments.
- A typical class alternates among lecture, collaborative worksheet work, and further lecture.

A brief historical perspective

ca. 300 BCE	**1700–1900**	**ca. 1900**
Euclid's *Elements* axiomatic plane geometry	major new developments in mathematics	set theory becomes a common foundation

- Euclid's geometry gave a powerful axiomatic account of plane geometry and a geometric way to think about number.
- Complex numbers, non-Euclidean geometry, infinite series, and calculus strained that older foundation.
- Cantor, Dedekind, and Hilbert helped make set theory a common language for modern mathematics.

Why set theory?

Since around 1900, set theory has been the basic foundational structure of mathematics. It provides a common language for discussing numbers, functions, and the objects on which calculus acts.

Working definition

A set is an unordered collection of objects, called its elements. A set is determined by the elements it contains.

Therefore, order and repetition do not matter:

$$\{1,2,3\} = \{3,2,1\}\,, \qquad \{1,1,2\} = \{1,2\}\,.$$

Membership and standard sets

Membership notation

$$a \in A \quad \text{means that } a \text{ is an element of } A, \qquad a \notin A \quad \text{means that it is not.}$$

$$1 \in \{1, 2, 3\}, \qquad 4 \notin \{1, 2, 3\}.$$

Standard sets

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}, \qquad \mathbb{R} = \text{the set of real numbers.}$$

Set-builder notation

The notation

$$\{n \in \mathbb{N} \mid n \text{ is even}\}$$

means the set of natural numbers n such that n is even. The vertical bar is read "such that."

For example,

$$\{n \in \mathbb{N} \mid n \text{ is odd and } 1 \leq n \leq 9\} = \{1, 3, 5, 7, 9\}.$$

Operations on sets

If A and B are sets, then

$$\begin{aligned} A \cup B &= \{x \mid x \in A \text{ or } x \in B\}, && \text{(union)}, \\ A \cap B &= \{x \mid x \in A \text{ and } x \in B\}, && \text{(intersection)}, \\ A \setminus B = A - B &= \{x \mid x \in A \text{ and } x \notin B\}, && \text{(difference)}. \end{aligned}$$

The set $A \setminus B$ makes sense whether or not $B \subseteq A$.

Subsets and equality

Subset

$$A \subseteq B \quad \Longleftrightarrow \quad \text{every element of } A \text{ is an element of } B.$$

Equivalently,

$$x \in A \Longrightarrow x \in B.$$

The fundamental criterion for equality of sets

$$A = B \quad \Longleftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

To prove that two sets are equal, prove both containments.

Functions

If X and Y are sets, a function

$$f\colon X \longrightarrow Y$$

is a rule which associates to every $x \in X$ a uniquely determined element $f(x) \in Y$.

Images and inverse images

Images and inverse images

If $S \subseteq X$, then

$$f(S) = \{f(s) \mid s \in S\}.$$

If $T \subseteq Y$, then

$$f^{-1}(T) = \{x \in X \mid f(x) \in T\}.$$

For $y \in Y$, we write

$$f^{-1}(y) = f^{-1}(\{y\}) = \{x \in X \mid f(x) = y\}.$$

Example: $f(x) = x^2$

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

$$f([0,1]) = [0,1],$$
$$f^{-1}([0,1]) = [-1,1],$$
$$f^{-1}(4) = f^{-1}(\{4\}) = \{-2, 2\}.$$

Proposition

Let $f: X \to Y$, and let $A, B \subseteq Y$. Then

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

To prove equality, prove both inclusions:

$$\begin{aligned} f^{-1}(A \cup B) &\subseteq f^{-1}(A) \cup f^{-1}(B), \\ f^{-1}(A) \cup f^{-1}(B) &\subseteq f^{-1}(A \cup B). \end{aligned}$$

Proof: first containment

Suppose that $x \in f^{-1}(A \cup B)$. Then

$$f(x) \in A \cup B.$$

Therefore,

$$f(x) \in A \quad \text{or} \quad f(x) \in B.$$

Thus,

$$x \in f^{-1}(A) \quad \text{or} \quad x \in f^{-1}(B),$$

which says

$$x \in f^{-1}(A) \cup f^{-1}(B).$$

Hence

$$f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B).$$

Proof: second containment

Suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Then

$$x \in f^{-1}(A) \quad \text{or} \quad x \in f^{-1}(B).$$

Therefore,

$$f(x) \in A \quad \text{or} \quad f(x) \in B.$$

Thus,

$$f(x) \in A \cup B,$$

so

$$x \in f^{-1}(A \cup B).$$

Hence

$$f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B).$$

Conclusion

Since both containments hold,

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$