

MATH 1610 / HONORS CALCULUS

Honors Calculus 1610

Lecture 10: The Rules of Differentiation

October 16, 2026

Today

- ♦ A warning about limits that do not exist.
- ♦ The rules: sums, scalar multiples, Leibniz, the chain rule.
- ♦ The lemma the product rule secretly needs: **differentiable** $\implies$ **continuous**.
- ♦ Powers, reciprocals, quotients.

Recall: the derivative

Definition

Let $f: (a, b) \rightarrow \mathbb{R}$ and $c \in (a, b)$. We say f is **differentiable at $x = c$** if

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

exists, and in that case we write $f'(c)$ for its value.

The derivative as a function

If the derivative exists at **every** $c \in (a, b)$, then we get a new function

$$f' : (a, b) \longrightarrow \mathbb{R}, \quad c \longmapsto f'(c),$$

and *this* function is what we call the derivative of f .

Two ways of writing the same thing, which we will use interchangeably:

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \quad \text{and} \quad f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c},$$

related by the substitution $h = x - c$.

An aside: a question about limits

Let f and g be defined on (a, b) and let $c \in (a, b)$.

Question

Is it true that

$$\lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) ?$$

Careful. This is a trick question.

The answer

What the limit law actually says is a **one-way** statement:

*if the two limits on the right exist,
then the one on the left exists and they are equal.*

It does not say the left side exists on its own. For example, neither $\lim_{x \rightarrow 0} \frac{1}{x}$ nor $\lim_{x \rightarrow 0} \frac{-1}{x}$ is a real number, and yet

$$\lim_{x \rightarrow 0} \left(\frac{1}{x} + \left(-\frac{1}{x} \right) \right) = \lim_{x \rightarrow 0} 0 = 0.$$

The equation is a statement about limits **that all exist**.

Sums and scalar multiples

Proposition

If f and g are differentiable at $x = c$ and $\lambda \in \mathbb{R}$, then $f + g$ and λf are differentiable at $x = c$, and

$$(f + g)'(c) = f'(c) + g'(c), \quad (\lambda f)'(c) = \lambda f'(c).$$

Both follow from the corresponding limit laws, once we notice that the difference quotient of a sum is the sum of the difference quotients.

Proof of the sum rule

$$\frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}.$$

As $h \rightarrow 0$ the two pieces on the right tend to $f'(x)$ and $g'(x)$, both of which exist by hypothesis. So the limit law applies, the left side has a limit, and

$$(f + g)'(x) = f'(x) + g'(x). \quad \square$$

Note where the aside just now was earning its keep.

The Leibniz rule

Proposition (product rule)

If $f'(x)$ and $g'(x)$ exist, then

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

By definition we must compute

$$\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h},$$

and the numerator is not obviously either difference quotient.

Add and subtract a middle term

$$\begin{aligned} & \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ = & \frac{f(x+h)g(x+h) - \textcolor{brown}{f(x+h)g(x)} + \textcolor{brown}{f(x+h)g(x)} - f(x)g(x)}{h} \\ = & f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} \cdot g(x). \end{aligned}$$

The same move as in the proof of the product law for limits.

Splitting the limit

The second term tends to $f'(x)g(x)$. For the first,

$$\lim_{h \rightarrow 0} f(x+h) \cdot \frac{g(x+h) - g(x)}{h} = \left(\lim_{h \rightarrow 0} f(x+h) \right) \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right),$$

provided both exist. The second is $g'(x)$.

And the first? We need

$$\lim_{h \rightarrow 0} f(x+h) = f(x),$$

which is to say we need f to be **continuous** at x . We only assumed it was differentiable.

An extra lemma is needed

Lemma

If f is differentiable at $x = c$, then f is continuous at $x = c$.

With the lemma in hand, $f(x + h) \rightarrow f(x)$, the first factor is $f(x)$, and the Leibniz rule follows:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x). \quad \square$$

Proof of the lemma

Let $\epsilon > 0$. We want $\delta > 0$ so that $0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.

Since $f'(c)$ exists, there is $\delta_1 > 0$ with

$$|h| < \delta_1 \implies \left| \frac{f(c+h) - f(c)}{h} - f'(c) \right| < \frac{\epsilon}{2}.$$

Put

$$\delta = \min \left\{ 1, \delta_1, \frac{\epsilon}{2(|f'(c)| + 1)} \right\}.$$

(The $+1$ is only so that this still makes sense when $f'(c) = 0$.)

Proof of the lemma, continued

Multiplying the displayed inequality by $|h|$,

$$|f(c+h) - f(c) - f'(c)h| < \frac{\epsilon|h|}{2} \leq \frac{\epsilon}{2}, \quad \text{since } |h| < \delta \leq 1.$$

Now take $0 < |x - c| < \delta$ and set $h = x - c$, so $x = c + h$ and $|h| < \delta \leq \delta_1$. Then

$$\begin{aligned} |f(x) - f(c)| &= |f(c+h) - f(c) - f'(c)h + f'(c)h| \\ &\leq |f(c+h) - f(c) - f'(c)h| + |f'(c)h| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square \end{aligned}$$

The chain rule

Proposition

Suppose g is differentiable at $x = c$, and f is differentiable at $x = b = g(c)$. Then $f \circ g$ is differentiable at $x = c$ and

$$(f \circ g)'(c) = f'(g(c)) g'(c).$$

The argument one wants to give

$$\lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{x - c} = \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c},$$

and then, writing $b = g(c)$ and $k = g(x) - g(c)$, recognise the first factor as

$$\frac{f(b+k) - f(b)}{k} \longrightarrow f'(b).$$

Why it is not allowed

We divided by $g(x) - g(c)$, and **that can be zero** for x arbitrarily close to c .

The simplest example is g constant, where it is zero for every x . A less degenerate one is $g(x) = x^2 \sin(1/x)$.

So the first factor need not even be defined, let alone have a limit. The repair is to replace it by something that **is** defined when the denominator vanishes.

The repair

For y near b , define

$$q(y) = \begin{cases} \frac{f(y) - f(b)}{y - b}, & y \neq b, \\ f'(b), & y = b. \end{cases}$$

Then

$$\lim_{y \rightarrow b} q(y) = f'(b) = q(b),$$

so q is continuous at $y = b$. This is the one and only place differentiability of f at b is used.

The identity that makes it work

$$(y - b) q(y) = f(y) - f(b) \quad \text{for every } y \text{ near } b.$$

For $y \neq b$ this is just clearing the denominator. For $y = b$ both sides are 0 — and that is exactly the case the naive argument could not handle.

Substituting $y = g(x)$:

$$f(g(x)) - f(g(c)) = q(g(x)) (g(x) - g(c)),$$

with no case distinction and nothing undefined.

Taking the limit

Divide by $x - c$ (legitimate: $x \neq c$ throughout the limit):

$$\frac{f(g(x)) - f(g(c))}{x - c} = q(g(x)) \cdot \frac{g(x) - g(c)}{x - c}.$$

Each factor has a limit, so the product law applies:

- ◆ g is differentiable, hence continuous at c , so $g(x) \rightarrow g(c) = b$; and q is continuous at b , so we may **plug in**:

$$q(g(x)) \longrightarrow q(g(c)) = q(b) = f'(b) = f'(g(c)).$$

- ◆ The second factor tends to $g'(c)$, by definition.

The chain rule

$$(f \circ g)'(c) = f'(g(c)) g'(c). \quad \square$$

Worth noticing what the two continuity facts did: g continuous at c got us *to* b , and f continuous at b let us *evaluate* there.

Powers

Start from $(x)' = 1$, and run the Leibniz rule:

$$(x^2)' = (x \cdot x)' = x \cdot 1 + 1 \cdot x = 2x,$$

$$(x^3)' = (x \cdot x^2)' = x \cdot (2x) + 1 \cdot x^2 = 3x^2,$$

and in general, if $(x^{n-1})' = (n-1)x^{n-2}$, then

$$(x^n)' = (x \cdot x^{n-1})' = x \cdot (n-1)x^{n-2} + 1 \cdot x^{n-1} = nx^{n-1}.$$

So $(x^n)' = nx^{n-1}$ for every $n \geq 1$, by induction.

The derivative of $1/x$

$$\begin{aligned}\left(\frac{1}{x}\right)' &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h \rightarrow 0} \frac{-h}{hx(x+h)} = -\frac{1}{x^2},\end{aligned}$$

valid for $x \neq 0$.

Again the cancellation of h is legal precisely because $h \neq 0$.

The reciprocal rule

Write $r(x) = 1/x$, so $r'(x) = -1/x^2$. If $f(x) \neq 0$, then $1/f = r \circ f$, and the chain rule gives

$$\left(\frac{1}{f(x)}\right)' = r'(f(x)) f'(x) = -\frac{1}{f(x)^2} f'(x) = -\frac{f'(x)}{f(x)^2}.$$

No new work: one derivative computed from the definition, plus the chain rule, gives every reciprocal at once.

The quotient rule

Now combine with Leibniz, writing $f/g = f \cdot (1/g)$:

$$\left(\frac{f(x)}{g(x)}\right)' = f'(x) \frac{1}{g(x)} + f(x) \left(\frac{1}{g(x)}\right)' = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2},$$

so

$$\left(\frac{f}{g}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

for those who like mnemonics: $\frac{\text{HI}}{\text{HO}} \rightsquigarrow \frac{\text{HO } d\text{HI} - \text{HI } d\text{HO}}{\text{HO HO}}$

Where we are going

♦ If f has a local maximum at c , then $f'(c) = 0$.

♦ Rolle's theorem and the mean value theorem:

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{for some } c \in (a, b).$$

♦ Consequences: $f' > 0$ means increasing, $f' < 0$ decreasing, $f' = 0$ constant.

♦ Critical points, and the second derivative test: $f'(c) = 0$ with $f'' < 0$ makes f' decreasing through c , hence a local maximum.