

MATH 1610 / HONORS CALCULUS

Honors Calculus 1610

Lecture 11: Extrema, the Mean Value Theorem, and the FTC

October 19, 2026

Today

- ◆ Where extrema can be: interior extrema are critical points.
- ◆ Rolle's theorem and the mean value theorem.
- ◆ What the MVT buys: the shape of a graph from the sign of f' .
- ◆ And then the payoff: the fundamental theorem of calculus.

Local extrema

Definition

Let $f: (a, b) \rightarrow \mathbb{R}$ and $c \in (a, b)$. We say c is a **local maximum** for f if there is $h > 0$ such that

$$f(c) \geq f(x) \quad \text{for all } x \in (c - h, c + h) \cap (a, b),$$

and a **local minimum** if instead $f(c) \leq f(x)$ there.

Local: we only compare $f(c)$ with nearby values, not with all of them.

Interior extrema

Proposition

Suppose $f: (a, b) \rightarrow \mathbb{R}$ has a local maximum at $c \in (a, b)$ and that $f'(c)$ exists. Then $f'(c) = 0$.

The proof uses the same device as the chain rule. Set

$$q(x) = \begin{cases} \frac{f(x) - f(c)}{x - c}, & x \neq c, \\ f'(c), & x = c, \end{cases}$$

so that $\lim_{x \rightarrow c} q(x) = f'(c) = q(c)$: that is, q is continuous at c . This is just a restatement of what it means for $f'(c)$ to exist.

Proof

Suppose $f'(c) \neq 0$, say $q(c) = f'(c) > 0$.

By the **sign preservation lemma** — continuity at c with $\epsilon = |q(c)|$ — there is $h > 0$ with

$$q(x) > 0 \quad \text{for all } x \in (c - h, c + h).$$

Pick $d \in (c, c + h)$. Then

$$\frac{f(d) - f(c)}{d - c} = q(d) > 0, \quad d - c > 0,$$

so $f(d) > f(c)$, and c is not a maximum. If instead $f'(c) < 0$, take $d \in (c - h, c)$: now $d - c < 0$ and $q(d) < 0$, so again $f(d) > f(c)$. $\square$

Three cautions

- ◆ **Interior** matters. On $[0, 1]$, $f(x) = x$ has a maximum at 1 and $f'(1) = 1 \neq 0$. The endpoint is not interior.
- ◆ **$f'(c)$ must exist.** $f(x) = -|x|$ has a maximum at 0 and no derivative there.
- ◆ The converse is **false**. $f(x) = x^3$ on $(-1, 1)$ has $f'(0) = 0$ and no extremum at 0.

Rolle's theorem

Theorem

If f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Note the asymmetry: continuity on the **closed** interval, differentiability only on the **open** one. The first is what the extreme value theorem needs.

Proof of Rolle

By the extreme value theorem f attains a minimum value m and a maximum value M on $[a, b]$.

- ◆ If either is attained at an **interior** point c , the proposition gives $f'(c) = 0$ and we are done.
- ◆ Otherwise both are attained only at the endpoints. Since $f(a) = f(b)$, that forces $m = M$, so f is **constant** — and then $f'(c) = 0$ for every $c \in (a, b)$.



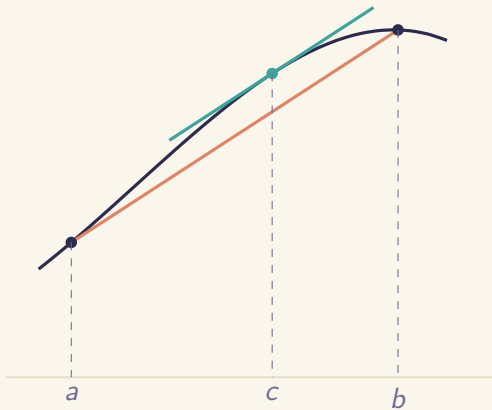
The mean value theorem

Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) , then there is $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The picture: some tangent is parallel to the chord



Proof of the mean value theorem

Let $m = \frac{f(b) - f(a)}{b - a}$ be the slope of the chord, and set

$$g(x) = f(x) - mx.$$

Then g is continuous on $[a, b]$ and differentiable on (a, b) , and

$$g(b) - g(a) = f(b) - f(a) - m(b - a) = 0,$$

so $g(a) = g(b)$.

By Rolle there is $c \in (a, b)$ with $g'(c) = 0$, that is $f'(c) - m = 0$. □

Graphing rules

Proposition

Let f be continuous on $[a, b]$ and differentiable on (a, b) .

1. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.
2. If $f'(x) < 0$ throughout, then f is strictly decreasing.
3. If $f'(x) = 0$ throughout, then f is constant.

One application of the MVT does all three

Take $x_1 < x_2$ in $[a, b]$. Then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) , so the MVT gives some $c \in (x_1, x_2)$ with

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \quad \text{i.e.} \quad f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

The factor $x_2 - x_1$ is **positive**, so the sign of $f(x_2) - f(x_1)$ is exactly the sign of $f'(c)$. All three statements follow at once. $\square$

What this replaces

In Lecture 8 we showed x^n is increasing on $\mathbb{R}^+$ by manipulating inequalities, and on Homework 5 you showed a sum of increasing functions is increasing the same way.

All of that is now a single line: differentiate and look at the sign.

This is the first place where the derivative earns its keep as a **tool** rather than an object of study.

Critical points

Definition

If $f: (a, b) \rightarrow \mathbb{R}$ is differentiable, we say $c \in (a, b)$ is a **critical point** if $f'(c) = 0$.

So the proposition says: a local extremum is a critical point.

The converse fails — $f(x) = x^3$ on $(-1, 1)$ has $f'(0) = 0$ and no extremum. So critical points are **candidates**, and we need a way to sort them.

The first derivative test

Let c be a critical point, and suppose there is an interval $(c - h, c + h) \subseteq (a, b)$ on which

f is increasing on $(c - h, c)$ and f is decreasing on $(c, c + h)$.

Then f has a local **maximum** at c . Swap the two words and you get a local minimum.

More generally: a critical point is a local extremum as soon as f' **changes sign** there.

The second derivative test

Proposition

Let $f: (a, b) \rightarrow \mathbb{R}$ be twice differentiable, let $c \in (a, b)$ be a critical point, and suppose f'' is continuous at c with $f''(c) \neq 0$. Then c is a local extremum — a minimum if $f''(c) > 0$, a maximum if $f''(c) < 0$.

Correction: the continuity of f'' was accidentally left out when this was stated in lecture. It is needed — see the remark after the proof.

We want to conclude that f' changes sign at c , and then quote the first derivative test.

Proof of the second derivative test

Say $f''(c) > 0$.

Since f'' is continuous at c , the sign preservation lemma gives an $h > 0$ with

$$f''(x) > 0 \quad \text{for all } x \in (c - h, c + h).$$

By the graphing rules applied to f' , this makes f' strictly increasing on $(c - h, c + h)$. But $f'(c) = 0$, so

$$f'(x) < 0 \text{ on } (c - h, c), \quad f'(x) > 0 \text{ on } (c, c + h).$$

So f decreases and then increases: a local minimum. $\square$

Where the continuity was needed

The one place: to get from $f''(c) > 0$ at the **single point** c to $f'' > 0$ on a whole **interval** around c .

That step is the sign preservation lemma, and that lemma is about **continuous** functions. Knowing only that the number $f''(c)$ is positive says nothing about f'' at nearby points.

And this is not an idle worry: a function can be differentiable everywhere and still have a derivative that is not continuous. You will build one on the homework.

The fundamental theorem

The fundamental theorem of calculus, I

Theorem

Let $f: [a, b] \rightarrow \mathbb{R}$, and suppose that for each $x \in [a, b]$ the integral

$$F(x) = \int_a^x f(t) dt$$

exists. If f is **continuous at c** , then F is differentiable at c and

$$F'(c) = f(c).$$

Differentiation undoes integration.

The difference quotient

$$\frac{F(c+h) - F(c)}{h} = \frac{\int_a^{c+h} f(t) dt - \int_a^c f(t) dt}{h} = \frac{1}{h} \int_c^{c+h} f(t) dt,$$

using additivity of the integral.

We want this to tend to $f(c)$. The trick is to put $f(c)$ into the integrand and show the rest is small:

$$\frac{1}{h} \int_c^{c+h} \left(f(c) + (f(t) - f(c)) \right) dt.$$

The main term

Split the integral in two (legitimate once we know both pieces converge):

$$\frac{1}{h} \int_c^{c+h} f(c) dt + \frac{1}{h} \int_c^{c+h} (f(t) - f(c)) dt.$$

The first integrand is a **constant** — a step function — so

$$\frac{1}{h} \int_c^{c+h} f(c) dt = \frac{1}{h} \cdot f(c) \cdot h = f(c),$$

with no limit needed at all.

The error term

Claim

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_c^{c+h} (f(t) - f(c)) dt = 0.$$

Let $\epsilon > 0$. By continuity of f at c there is $\delta > 0$ with $|t - c| < \delta \implies |f(t) - f(c)| < \epsilon$. So for $0 < |h| < \delta$ the integrand is bounded by ϵ on the whole interval of integration, and

$$\left| \frac{1}{h} \int_c^{c+h} (f(t) - f(c)) dt \right| \leq \frac{1}{|h|} \cdot \epsilon \cdot |h| = \epsilon.$$

Hence $F'(c) = f(c)$. $\square$

The zero derivative theorem

Theorem

If f is continuous on $[a, b]$, differentiable on (a, b) , and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

This is case 3 of the graphing rules, and it is the MVT again: for any $x_1 < x_2$, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$.

Corollary

If $F' = G'$ on an interval, then $F - G$ is constant: $F = G + C$.

Antiderivatives

Definition

P is an **antiderivative** (or *primitive*) of f on $[a, b]$ if $P' = f$ there.

By the corollary, any two antiderivatives of the same function differ by a constant — so “the” antiderivative is well defined up to one number.

FTC I says that a continuous f **has** one, namely $F(x) = \int_a^x f$.

The fundamental theorem of calculus, II

Theorem

Let f be continuous and let P be any antiderivative of f . Then for every c ,

$$P(x) = P(c) + \int_c^x f(t) dt.$$

Proof. By FTC I the right-hand side has derivative $f(x)$; by assumption so does P . Two functions with the same derivative differ by a constant. Setting $x = c$ makes both sides equal to $P(c)$, since $\int_c^c f = 0$ — so the constant is 0. $\square$

The evaluation rule

Take $c = a$ and $x = b$:

$$\int_a^b f(t) dt = G(b) - G(a) = \left[G(x) \right]_a^b$$

for **any** antiderivative G of f .

This is the computational payoff of the whole course so far: to integrate f , do not go back to step functions and suprema — **run the derivative rules backwards**.

Next time

A change of subject: vectors.

Still owed, and we will come back to them:

- ◆ $|x|$ is continuous but not differentiable at 0: one-sided limits.
- ◆ Higher derivatives and notation.
- ◆ Trig identities and $\frac{\sin x}{x} \rightarrow 1$, which is what we need before differentiating sin and cos.