

Honors Calculus 1610

Lecture 12: Vectors, the Dot Product, and Vector-Valued Limits

October 23, 2026

Today

- ◆ Vectors in $\mathbb{R}^n$: addition and scalar multiplication.
- ◆ The dot product, length, and angles.
- ◆ Two useful inequalities: Cauchy–Schwarz and the triangle inequality.
- ◆ Back to calculus: limits of vector-valued functions.

$\mathbb{R}^n$

Definition

$$\mathbb{R}^n = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{R}\} = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ times}}.$$

Elements of $\mathbb{R}^n$ are called (n -dimensional) **vectors**, or n -vectors.

$\mathbb{R}^1$ is the line, $\mathbb{R}^2$ the plane, $\mathbb{R}^3$ space: the point $(2, 3) \in \mathbb{R}^2$ is two over and three up.

Notation. A letter with an arrow is a vector, $\vec{a} = (a_1, \dots, a_n)$. We may leave the arrow off.

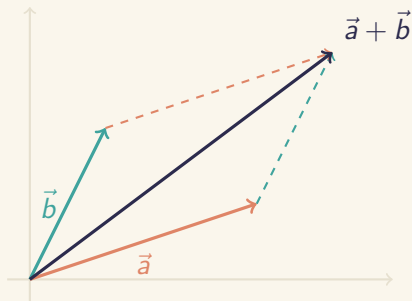
Vector addition

Definition

For $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$, $\vec{a} + \vec{b} = (a_1 + b_1, \dots, a_n + b_n)$.

Geometrically: put the tail of $\vec{b}$ at the head of $\vec{a}$.

The parallelogram law



Going along $\vec{a}$ then $\vec{b}$, or along $\vec{b}$ then $\vec{a}$, lands at the same point.

Scalar multiplication

Definition

For $\lambda \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^n$, $\lambda \vec{a} = (\lambda a_1, \dots, \lambda a_n)$.

This **stretches** $\vec{a}$ by a factor of λ (and reverses it if $\lambda < 0$).

Standard identities. For vectors $\vec{a}, \vec{b}, \vec{c}$ and scalars λ, μ :

1. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
2. $\lambda(\mu \vec{a}) = (\lambda\mu) \vec{a}$
3. $(\lambda + \mu) \vec{a} = \lambda \vec{a} + \mu \vec{a}$
4. $\lambda(\vec{a} + \vec{b}) = \lambda \vec{a} + \lambda \vec{b}$

Why the identities hold

Each one is checked **coordinate by coordinate**, where it reduces to a law about real numbers. For example, identity 2:

$$\lambda(\mu\vec{a}) = \lambda(\mu a_1, \dots, \mu a_n) = (\lambda(\mu a_1), \dots, \lambda(\mu a_n)),$$

$$(\lambda\mu)\vec{a} = ((\lambda\mu)a_1, \dots, (\lambda\mu)a_n),$$

and these agree because $\lambda(\mu a_i) = (\lambda\mu)a_i$ for each i : associativity of multiplication in $\mathbb{R}$.

(On the board identity 3 was written with $\mu\vec{b}$ on the right; it should be $\mu\vec{a}$.)

The dot product

Definition

For $\vec{a}, \vec{b} \in \mathbb{R}^n$, the **dot product** is the real number

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^n a_i b_i.$$

Geometrically, it measures “how in line with each other” two vectors are:

- ♦ $+$: pointing in similar directions,
- ♦ $-$: pointing in opposite directions,
- ♦ 0 : perpendicular.

Dot product identities

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$
3. $(\lambda \vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b})$
4. $\vec{a} \cdot \vec{a} \geq 0$, and $\vec{a} \cdot \vec{a} = 0$ if and only if $\vec{a} = \vec{0}$.

Proof of 4. $\vec{a} \cdot \vec{a} = a_1^2 + \cdots + a_n^2$, and each $a_i^2 \geq 0$, so the sum is ≥ 0 . If any $a_i \neq 0$ then $a_i^2 > 0$ and the sum is positive. So if the sum is 0, every $a_i^2 = 0$, hence every $a_i = 0$. $\square$

Length

Definition

$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$, the **length** of $\vec{a}$.

This makes sense because $\vec{a} \cdot \vec{a} \geq 0$.

In $\mathbb{R}^2$, $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$ is exactly the Pythagorean length of the arrow.

In $\mathbb{R}^3$, Pythagoras twice: $(a_1, a_2, 0)$ is at distance $\sqrt{a_1^2 + a_2^2}$ from the origin, and climbing a_3 more gives

$$\sqrt{(\sqrt{a_1^2 + a_2^2})^2 + a_3^2} = \sqrt{a_1^2 + a_2^2 + a_3^2} = \|\vec{a}\|.$$

Cauchy–Schwarz

Theorem

For vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$,

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|,$$

with equality if and only if one of $\vec{a}, \vec{b}$ is a scalar multiple of the other.

Correction: on the board the equality case read “iff $\vec{a} = \lambda \vec{b}$ for some λ ,” which fails when $\vec{b} = \vec{0} \neq \vec{a}$: equality holds, but $\vec{a}$ is not a multiple of $\vec{0}$. “One is a multiple of the other” covers it: $\vec{b} = 0 \cdot \vec{a}$.

Proof, case $\vec{b} = \vec{0}$. Both sides are 0. ✓

Proof, case $\vec{b} \neq \vec{0}$

Then $\|\vec{b}\| \neq 0$, so we may divide by it. For any $t \in \mathbb{R}$,

$$0 \leq \|\vec{a} - t\vec{b}\|^2 = (\vec{a} - t\vec{b}) \cdot (\vec{a} - t\vec{b}) = \|\vec{a}\|^2 - 2t(\vec{a} \cdot \vec{b}) + t^2\|\vec{b}\|^2.$$

Set $t = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}$:

$$0 \leq \|\vec{a}\|^2 + \frac{(\vec{a} \cdot \vec{b})^2}{\|\vec{b}\|^2} - 2\frac{(\vec{a} \cdot \vec{b})^2}{\|\vec{b}\|^2} = \|\vec{a}\|^2 - \frac{(\vec{a} \cdot \vec{b})^2}{\|\vec{b}\|^2}.$$

So $(\vec{a} \cdot \vec{b})^2 \leq \|\vec{a}\|^2\|\vec{b}\|^2$, and taking square roots, $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$.

The equality case

If $|\vec{a} \cdot \vec{b}| = \|\vec{a}\| \|\vec{b}\|$, then every inequality on the previous slide is an equality, so

$$\|\vec{a} - t\vec{b}\| = 0 \quad \text{for } t = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}.$$

By identity 4, $\vec{a} - t\vec{b} = \vec{0}$, that is, $\vec{a} = t\vec{b}$.

Conversely, if one vector is a multiple of the other, both sides are easily seen to agree.

□

The triangle inequality

Theorem

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|.$$

Proof.

$$\begin{aligned}\|\vec{a} + \vec{b}\|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a} \cdot \vec{b} \\ &\leq \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\|\vec{a}\|\|\vec{b}\| = (\|\vec{a}\| + \|\vec{b}\|)^2,\end{aligned}$$

using Cauchy–Schwarz in the middle. Both sides are nonnegative, so take square roots. $\square$

The picture: the side $\vec{a} + \vec{b}$ of the triangle is no longer than the other two sides together.

Angles

For nonzero $\vec{a}, \vec{b}$, Cauchy–Schwarz says

$$-1 \leq \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \leq 1.$$

Definition

The **angle** between nonzero vectors $\vec{a}$ and $\vec{b}$ is the number $\theta \in [0, \pi]$ with

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}.$$

Cauchy–Schwarz is exactly what makes this definition legitimate.

Back to calculus

Vector-valued functions

Definition

If X is a set, a **vector-valued function** on X is a function $\vec{f}: X \rightarrow \mathbb{R}^n$.

For each $x \in X$, $\vec{f}(x) \in \mathbb{R}^n$ has n coordinates, so $\vec{f}$ amounts to n ordinary functions $f_i: X \rightarrow \mathbb{R}$:

$$\vec{f}(x) = (f_1(x), \dots, f_n(x)), \quad \vec{f} = (f_1, \dots, f_n).$$

Example. $\vec{f}: [0, 1] \rightarrow \mathbb{R}^2$, $\vec{f}(t) = (t^2, 1 + t)$. Then $\vec{f}(0) = (0, 1)$ and $\vec{f}(1) = (1, 2)$: as t runs over $[0, 1]$, $\vec{f}(t)$ traces a curve in the plane.

Limits of vector-valued functions

Definition

Let $\vec{f}: (a, b) \rightarrow \mathbb{R}^n$ and $c \in (a, b)$. We write

$$\lim_{t \rightarrow c} \vec{f}(t) = \vec{L} \in \mathbb{R}^n$$

to mean: for every $\epsilon > 0$ there is $\delta > 0$ such that

$$0 < |t - c| < \delta \implies \|\vec{f}(t) - \vec{L}\| < \epsilon.$$

This is the definition we have used all along, with the absolute value on the right replaced by the **length**.

Example

$$\vec{f}(t) = \left(\frac{t^2 - 1}{t - 1}, 2t + 3 \right), \quad \text{claim: } \lim_{t \rightarrow 1} \vec{f}(t) = (2, 5).$$

Given $\epsilon > 0$, choose $\delta = \epsilon/5$, and let $0 < |t - 1| < \delta$. Since $t \neq 1$, $\frac{t^2 - 1}{t - 1} = t + 1$, so the two coordinate differences are

$$D_1 = |(t + 1) - 2| = |t - 1| < \frac{\epsilon}{5}, \quad D_2 = |(2t + 3) - 5| = 2|t - 1| < \frac{2\epsilon}{5}.$$

Therefore

$$\|\vec{f}(t) - (2, 5)\| = \sqrt{D_1^2 + D_2^2} < \sqrt{\frac{\epsilon^2}{25} + \frac{4\epsilon^2}{25}} = \epsilon \sqrt{\frac{5}{25}} < \epsilon.$$

Next time

- ◆ Limits of vector-valued functions are computed **componentwise**.
- ◆ Derivatives of vector-valued functions: velocity and speed.
- ◆ Orthogonality and projections.
- ◆ Lines, planes, and the cross product.