Honors Calculus 1610

Lecture 17: Reparametrization, Gradient Fields, and Second Derivatives

November 16, 2026

Today

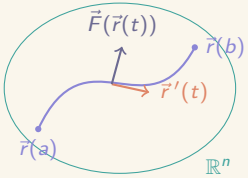
- Line integrals do not depend on the parametrization: a proof by u-substitution.
- The notation $\int_C \vec{F} \cdot d\vec{r}$.
- For a gradient field, the line integral depends only on the endpoints.
- When does ∇f exist? Directional derivatives again.
- C^1, C^2, ...; mixed partials are symmetric.
- Symmetric and antisymmetric matrices; the curl of a vector field.

Recall: line integrals

A curve $\vec{r}\colon [a,b] \to \mathbb{R}^n$ and a vector field $\vec{F}\colon \mathbb{R}^n \to \mathbb{R}^n$. At time t we have two vectors:

$$\vec{r}\,'(t) \qquad \text{and} \qquad \vec{F}(\vec{r}(t)),$$

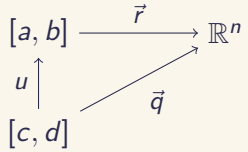
and their dot product is a scalar. The line integral is $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$.



Example: $\vec{F}$ is a force and $\vec{r}(t)$ is the position at time t; this is the work done along the path.

Fact. This integral depends only on the path in $\mathbb{R}^n$ and the direction it is traced, not on the parametrization.

Reparametrization



$u(c) = a, \quad u(d) = b, \qquad \vec{q}(s) = \vec{r}(u(s)).$

$\vec{r}$ and $\vec{q}$ trace out the same path, with the same starting point $\vec{r}(a) = \vec{q}(c)$ and the same end point $\vec{r}(b) = \vec{q}(d)$. This is one meaning of reparametrization.

Proposition

Let $\vec{r}$ and u have continuous derivatives, $\vec{F}$ be continuous, and $\vec{q} = \vec{r} \circ u$ as above. Then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_c^d \vec{F}(\vec{q}(s)) \cdot \vec{q}\,'(s)\, ds.$$

Recall: u-substitution

Lemma

Suppose $u\colon [c,d] \to [a,b]$ is such that u' exists and is continuous, and $g\colon [a,b] \to \mathbb{R}$ is continuous. Then $\displaystyle\int_c^d g(u(s))\,u'(s)\,ds = \int_{u(c)}^{u(d)} g(t)\,dt$.

Proof. Let $H(x) = \int_a^x g(t)\,dt$. By FTC I, $H'(x) = g(x)$. By the chain rule, $(H \circ u)'(s) = H'(u(s))\,u'(s) = g(u(s))\,u'(s)$, which is continuous. By FTC II (twice),

$$\int_c^d g(u(s))u'(s)\,ds = (H\circ u)(d) - (H\circ u)(c) = H(u(d)) - H(u(c))$$

$$= \int_{u(c)}^{u(d)} H'(t)\,dt = \int_{u(c)}^{u(d)} g(t)\,dt. \qquad \square$$

Proof of the proposition

By the chain rule (one variable, in each component),

$$\vec{q}\,'(s) = \vec{r}\,'(u(s))\,u'(s).$$

So

$$\int_c^d \vec{F}(\vec{q}(s)) \cdot \vec{q}\,'(s)\,ds = \int_c^d \underbrace{\vec{F}(\vec{r}(u(s))) \cdot \vec{r}\,'(u(s))}_{h(u(s))}\,u'(s)\,ds,$$

where $h(t) = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$, a continuous function on $[a, b]$. By u-substitution, this is

$$\int_{u(c)}^{u(d)} h(t)\,dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt. \qquad \square$$

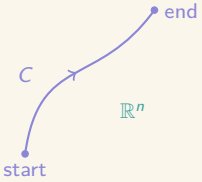
Notation

We write

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_C \vec{F} \cdot d\vec{r},$$

where C denotes the path in $\mathbb{R}^n$ that $\vec{r}$ runs through as t runs through $[a, b]$, with a specified orientation: a start and an end.

By the proposition, the right side makes sense: any parametrization of C (from start to end) gives the same number.



Gradient fields

Observation

Let $\vec{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ and suppose $\vec{F} = \nabla G$ for some $G\colon \mathbb{R}^n \to \mathbb{R}$ with ∇G continuous. Then $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of C:

$$\int_C \vec{F} \cdot d\vec{r} = G(\text{end point of } C) - G(\text{starting point of } C).$$



Proof. With $\vec{r}\colon [a,b] \to \mathbb{R}^n$ a parametrization of C,

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \nabla G(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_a^b (G \circ \vec{r})'(t)\, dt \overset{\text{FTC II}}{=} G(\vec{r}(b)) - G(\vec{r}(a)),$$

using the chain rule along a curve. □

When does the gradient exist?

Theorem

Let $S \subseteq \mathbb{R}^n$ be open and $f : S \to \mathbb{R}$. Suppose all the partial derivatives $\dfrac{\partial f}{\partial x_i}$ exist near $\vec{a} \in S$, and are continuous at $\vec{a}$. Then ∇f exists at $\vec{a}$: f is differentiable at $\vec{a}$.

Without the continuity, the partial derivatives can exist at $\vec{a}$ while f is not differentiable there.

Directional derivatives

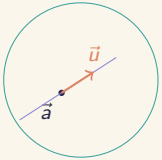
Definitions

$\vec{u} \in \mathbb{R}^n$ is a unit vector if $\|\vec{u}\| = 1$. The directional derivative of f at $\vec{a}$ in the direction $\vec{u}$ is

$$D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}.$$

Example: $\frac{\partial f}{\partial x_i}(\vec{a}) = D_{\vec{e}_i} f(\vec{a})$.

If ∇f exists at $\vec{a}$, then $\nabla f(\vec{a}) \cdot \vec{u} = D_{\vec{u}} f(\vec{a})$ (Lecture 16).



Intuition. ∇f points in the direction of fastest growth of f.

Higher derivatives

If $f\colon S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$, the partial derivatives $\frac{\partial f}{\partial x_i}\colon S \to \mathbb{R}$ are functions again (where they exist), so we can take their partial derivatives:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j\, \partial x_i}.$$

Definition

f is called C^0 if it is continuous; C^1 if all partial derivatives $\frac{\partial f}{\partial x_i}$ exist and are continuous; C^2 if all second partial derivatives exist and are continuous; ...; and C^∞, or smooth, if it is C^k for every k.

Mixed partials are symmetric

Theorem

If f is C^2, then

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j} = \frac{\partial^2 f}{\partial x_j \, \partial x_i} \qquad \text{for all } i, j.$$

So the second derivative information of $f : \mathbb{R}^n \to \mathbb{R}$,

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix},$$

is a symmetric matrix: unchanged by flipping across the diagonal.

Symmetric and antisymmetric matrices

The transpose A^T of a square matrix A is A flipped along the diagonal: $(A^T)_{ij} = A_{ji}$.

- A is symmetric if $A^T = A$, that is, $A - A^T = 0$.
- B is antisymmetric if $B^T = -B$.
- $\frac{1}{2}(A - A^T)$ is antisymmetric: the antisymmetrization of A. It is 0 exactly when A is symmetric.

An antisymmetric matrix has 0s on the diagonal, and the entries below the diagonal determine those above:

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \text{ (one number)}, \qquad \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix} \text{ (three numbers: a vector in } \mathbb{R}^3\text{)}.$$

The derivative matrix of a vector field

For a vector field $\vec{F} = (F_1, \ldots, F_n) \colon \mathbb{R}^n \to \mathbb{R}^n$, collect the partial derivatives in a matrix, with row i the gradient of F_i:

$$D\vec{F} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}, \qquad (D\vec{F})_{ij} = \frac{\partial F_i}{\partial x_j}.$$

When $\vec{F} = \nabla f$, the entries of $D\vec{F}$ are the second partial derivatives of f.

The curl of a vector field

Definition

The curl of $\vec{F}$ is the antisymmetric matrix $\operatorname{curl} \vec{F} = D\vec{F} - (D\vec{F})^T$, with entries $\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}$.

So $\operatorname{curl} \vec{F}$ is twice the antisymmetrization of $D\vec{F}$.

Correction: on the board, $D\vec{F}$ was written with rows and columns swapped, and the curl as $\frac{1}{2}\left((D\vec{F})^T - D\vec{F}\right)$. The convention above is the standard one, and with it the curl has no factor $\frac{1}{2}$. Apologies for the mix-up!