Honors Calculus 1610

Lecture 18: All the Derivatives, and Differential Forms

November 20, 2026

Today

- The goal: Stokes's theorem, $\int_R d\omega = \int_{\partial R} \omega$, and the theorems it contains.
- A brief accounting of all the derivatives so far: gradient, derivative matrix, Hessian, curl.
- Curl in $\mathbb{R}^3$, and a new one: the divergence.
- $\nabla \times (\nabla g) = 0$ and $\nabla \cdot (\nabla \times \vec{g}) = 0$.
- The formal calculus of forms: d, and wedges.

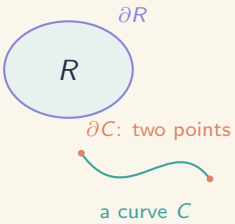
The goal: Stokes's theorem

Stokes's theorem (the goal; stated loosely)

If $R \subseteq \mathbb{R}^n$ is a d-dimensional region with (oriented) boundary ∂R, and ω is a $(d-1)$-form, then

$$\int_R d\omega = \int_{\partial R} \omega.$$

Here $d\omega$ is the exterior derivative of ω, a d-form.



Example: an interval

Let $R = [a, b] \subseteq \mathbb{R}$. Its boundary is two points with signs:

$$\partial R = \{a, b\}, \qquad a \text{ counted with } -, \quad b \text{ with } + \qquad \text{(oriented from } a \text{ to } b\text{)}.$$

A 0-form is a function g, and its integral over ∂R is $g(b) - g(a)$. If $f = g'$, then dg is the 1-form $f(t)\,dt$, and Stokes's theorem says

$$g(b) - g(a) = \int_{\partial R} g = \int_R dg = \int_a^b f(t)\,dt.$$

This is the fundamental theorem of calculus:

$$\int_a^b f(t)\,dt = g(b) - g(a) \qquad \text{when } g' = f.$$

Cheat sheet: the other cases

In $\mathbb{R}^n$, 0-forms are scalar functions, and so are n-forms.

- **Curves.** f a 1-form with $f = dg$, which is ∇g:

$$\int_C f = \int_C dg = \int_{\partial C} g = g(\text{end}) - g(\text{start}), \quad \text{i.e.} \quad \int_C \nabla g \cdot d\vec{r} = g(\text{end}) - g(\text{start}).$$

- **Surfaces.** f a 2-form with $f = \operatorname{curl} g$: $\displaystyle\int_S f = \int_{\partial S} g$, with ∂S the boundary curve of the surface S ("Green's theorem").
- **Solids.** f a 3-form with $f = \operatorname{div} g$: $\displaystyle\int_B f = \int_B \operatorname{div} g = \int_{\partial B} g$, with ∂B the boundary surface of the solid B (the "divergence theorem").

A brief accounting: the gradient

For $f\colon \mathbb{R}^n \to \mathbb{R}$, the gradient $\nabla f(\vec{x})$ (if it exists) is the vector with

$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{x}+\vec{h}) - f(\vec{x}) - \nabla f(\vec{x})\cdot\vec{h}}{\|\vec{h}\|} = 0.$$

That is, $f(\vec{x}+\vec{h}) \approx f(\vec{x}) + \nabla f(\vec{x})\cdot\vec{h}$, just as $f(x+h) \approx f(x) + f'(x)\,h$ in one variable, with $f'(x)$ the slope.

When it exists, we can also write it explicitly:

$$\nabla f(\vec{x}) = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\Big).$$

Secretly, the gradient is a 1-form

∇f corresponds to a covector, the linear function $\vec{h} \mapsto \nabla f \cdot \vec{h}$. A covector-valued function is a 1-form.

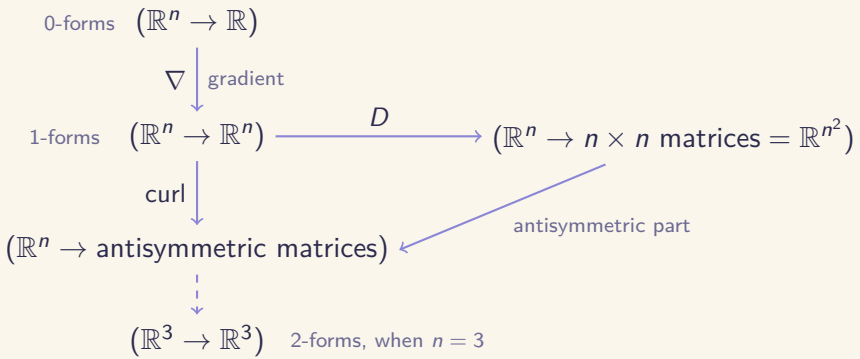
For $\vec{F}(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$, the line integral is also written

$$\int_C \vec{F} \cdot d\vec{r} = \int_C f_1 \, dx + f_2 \, dy + f_3 \, dz.$$

For a function f, the 1-form matching ∇f is

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

All the derivatives



$\operatorname{curl} \vec{F} = D\vec{F} - (D\vec{F})^T$, the antisymmetric matrix built from the derivative matrix.

The derivative matrix and the Hessian

For $\vec{f} = (f_1, \ldots, f_n) \colon \mathbb{R}^n \to \mathbb{R}^n$:

$$D\vec{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

If $\vec{f} = \nabla g$, this is the Hessian of g:

$$D(\nabla g) = H_g = \begin{pmatrix} \frac{\partial^2 g}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 g}{\partial x_n \partial x_1} \\ \vdots & & \vdots \\ \frac{\partial^2 g}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 g}{\partial x_n \partial x_n} \end{pmatrix}.$$

Recall

If g is C^2, then $\dfrac{\partial^2 g}{\partial x_i \partial x_j} = \dfrac{\partial^2 g}{\partial x_j \partial x_i}$. That is, for $\vec{f} = \nabla g$, $D\vec{f} = (D\vec{f})^T$.

Curl in $\mathbb{R}^3$

Mnemonic. For $\vec{f} = (f_1, f_2, f_3)$,

$$\nabla \times \vec{f} = \det \begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ f_1 & f_2 & f_3 \end{pmatrix} = \Big(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \Big).$$

These three numbers fill the antisymmetric matrix $D\vec{f} - (D\vec{f})^T$. Writing $\partial_j f_i$ for $\frac{\partial f_i}{\partial x_j}$,

$$D\vec{f} - (D\vec{f})^T = \begin{pmatrix} 0 & \partial_2 f_1 - \partial_1 f_2 & \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 & 0 & \partial_3 f_2 - \partial_2 f_3 \\ \partial_1 f_3 - \partial_3 f_1 & \partial_2 f_3 - \partial_3 f_2 & 0 \end{pmatrix}.$$

The diagonal cancels; entries opposite each other are negatives.

Divergence

$(n-1)$-forms
$(\mathbb{R}^n \to \mathbb{R}^n)$
$\downarrow \operatorname{div} = \nabla\cdot$
$(\mathbb{R}^n \to \mathbb{R})$
n-forms, "volume forms"

With $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right),$$

$$\nabla \times \vec{f} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \times (f_1, f_2, f_3),$$

$$\nabla \cdot \vec{f} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot (f_1, \ldots, f_n).$$

Definition

If $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^1, its divergence is the function

$$\operatorname{div} \vec{f} = \nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n}.$$

Two steps in a row give zero

Remark

If g is C^2, then

$$\nabla \times (\nabla g) = 0,$$

by the definition of curl and the symmetry of mixed partials.

Converse (stated). If $\nabla \times \vec{f} = 0$ for $\vec{f}$ defined in a ball, then $\vec{f} = \nabla g$ for some g.

Similarly, in $\mathbb{R}^3$

If $\vec{g}$ is C^2, then

$$\nabla \cdot (\nabla \times \vec{g}) = 0.$$

Converse (stated). If $\nabla \cdot \vec{f} = 0$ for $\vec{f}$ defined in a ball, then $\vec{f} = \nabla \times \vec{g}$ for some $\vec{g}$.

The formal calculus of forms

1-forms

A 1-form on $\mathbb{R}^n$ is an expression $f_1\,dg_1 + f_2\,dg_2 + \cdots + f_r\,dg_r$, where all the f_i and g_i are (smooth) functions, with the rules

$$d(fg) = f\,dg + g\,df, \qquad df = \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n.$$

1-forms in $\mathbb{R}^1$. Here $df = \frac{\partial f}{\partial x}\,dx$. So for $g = g(x)$,

$$f\,dg = f\,\frac{\partial g}{\partial x}\,dx, \qquad f_1\,dg_1 + f_2\,dg_2 = \Big(f_1\frac{\partial g_1}{\partial x} + f_2\frac{\partial g_2}{\partial x}\Big)dx.$$

Every 1-form in $\mathbb{R}^1$ is a function times dx.

Climbing with d

0-form	$f : \mathbb{R}^n \to \mathbb{R}$	$f \longmapsto df = \frac{\partial f}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n$
	$\downarrow d$	
1-form	$f\, dg$	
	$\downarrow d$	
2-form	$df \wedge dg$	

To say what $df \wedge dg$ means, we need wedges.

Wedges: the exterior (Grassmann) algebra

Definition

A Grassmann element is a symbol of the form

$$\vec{v}_1 \wedge \vec{v}_2 \wedge \vec{v}_3 + \vec{v}_4 \wedge \vec{v}_5$$

(sums of wedges of vectors), with the rules

$$\vec{v} \wedge \vec{w} = -\vec{w} \wedge \vec{v}, \qquad (\vec{v}_1 + \vec{v}_2) \wedge \vec{w} = \vec{v}_1 \wedge \vec{w} + \vec{v}_2 \wedge \vec{w}, \qquad \lambda(\vec{v} \wedge \vec{w}) = (\lambda\vec{v}) \wedge \vec{w}.$$

In particular $\vec{v} \wedge \vec{v} = -\vec{v} \wedge \vec{v}$, so $\vec{v} \wedge \vec{v} = 0$.

Example: wedges in $\mathbb{R}^2$

$$\begin{aligned}(a,b)\wedge(c,d) &= ((a,0)+(0,b))\wedge((c,0)+(0,d))\\ &= (a,0)\wedge(c,0)+(a,0)\wedge(0,d)+(0,b)\wedge(c,0)+(0,b)\wedge(0,d).\end{aligned}$$

The first term is $ac\,(1,0)\wedge(1,0)$, and $(1,0)\wedge(1,0) = -(1,0)\wedge(1,0)$, so it is 0. Likewise the last term is 0. The middle terms are

$$ad\,(1,0)\wedge(0,1) + bc\,(0,1)\wedge(1,0) = (ad-bc)\,(1,0)\wedge(0,1).$$

So

$$(a,b)\wedge(c,d) = (ad-bc)\,(1,0)\wedge(0,1).$$

Example: an exterior derivative

In $\mathbb{R}^2$, take the 1-form $\omega = x^2 y\,dx + xy\,dy$. Each term has the shape $f\,dg$, and $d(f\,dg) = df \wedge dg$:

$$\begin{aligned} d\omega &= d(x^2 y) \wedge dx + d(xy) \wedge dy \\ &= (2xy\,dx + x^2\,dy) \wedge dx + (y\,dx + x\,dy) \wedge dy \\ &= x^2\,dy \wedge dx + y\,dx \wedge dy \qquad (\text{since } dx \wedge dx = dy \wedge dy = 0) \\ &= (y - x^2)\,dx \wedge dy. \end{aligned}$$

In general, for $\omega = P\,dx + Q\,dy$ the same steps give $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the one number in the curl of (P, Q).

Example: $d(df) = 0$

In $\mathbb{R}^2$, take $f = x^2y^3$. Then $df = 2xy^3\,dx + 3x^2y^2\,dy$, and

$$\begin{aligned} d(df) &= d(2xy^3) \wedge dx + d(3x^2y^2) \wedge dy \\ &= (2y^3\,dx + 6xy^2\,dy) \wedge dx + (6xy^2\,dx + 6x^2y\,dy) \wedge dy \\ &= 6xy^2\,dy \wedge dx + 6xy^2\,dx \wedge dy = 0. \end{aligned}$$

The two surviving terms are the mixed partials $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = 6xy^2$, and $dy \wedge dx = -dx \wedge dy$ makes them cancel. This is $\nabla \times (\nabla f) = 0$ in the language of forms.