Honors Calculus 1610

Lecture 3: Order, Completeness, and Square Roots

September 4, 2026

Today

- Finish the field axioms and add the order axioms.
- See a field that cannot be ordered.
- Upper bounds, suprema, and the definition of a complete ordered field.
- Watch $\mathbb{Q}$ fail to be complete, and prove that $\sqrt{2} \notin \mathbb{Q}$.
- Prove that every nonnegative real number has a square root.

Recall: the field axioms

A field is a set F with addition and multiplication satisfying:

Commutative: $a + b = b + a, \quad ab = ba.$

Associative: $(ab)c = a(bc), \quad a + (b + c) = (a + b) + c.$

Distributive: $a(b + c) = ab + ac.$

Identities: there are elements $0, 1$ with

$$a + 0 = a, \qquad a \cdot 1 = a, \qquad \text{and} \qquad 0 \neq 1.$$

Recall: inverses

Inverses: given $a \in F$ there is $-a \in F$ with

$$a + (-a) = 0,$$

and if $a \neq 0$ there is $a^{-1} \in F$ with

$$aa^{-1} = 1.$$

These axioms say nothing about which elements are "positive," or about which elements are "large." That is the next layer of structure.

The order axioms

Suppose F is a field. We say that F is ordered if we have specified a subset

$$F^+ \subseteq F, \qquad \text{“the positive elements,”}$$

such that:

1. if $x, y \in F^+$, then $x + y \in F^+$ and $xy \in F^+$;
2. if $x \in F$, then exactly one of the following holds:

$$x \in F^+, \qquad -x \in F^+, \qquad x = 0;$$

3. $0 \notin F^+$.

Sidebar: the notation $<$

If F is an ordered field, we write

$$\boxed{a < b} \qquad \text{to mean} \qquad b - a \in F^+.$$

Similarly,

$$a > b \iff b < a,$$

and $a \leq b$ means $a < b$ or $a = b$.

In particular $a \in F^+$ exactly when $a > 0$.

A first proposition about order

Proposition

If $a, b, c \in F$ and $a < b$, then

$$a + c < b + c, \qquad \text{and} \qquad ac < bc \ \text{ if } c \in F^+.$$

Remember that we have *defined* $x < y$ to mean $y - x \in F^+$. So each statement is really a statement about which differences are positive, and the only tools we have are the field axioms and the two order axioms.

Proof: adding c to both sides

We are given $a < b$, which by definition says

$$b - a \in F^+.$$

We want $a + c < b + c$, which by definition says

$$(b + c) - (a + c) \in F^+.$$

Now simplify that difference using commutativity and associativity:

$$(b + c) - (a + c) = b + c - a - c = b - a.$$

That element is in F^+ by hypothesis, so $a + c < b + c$.

Proof: multiplying by a positive c

Now suppose in addition that $c \in F^+$, that is, $c > 0$.

We want $ac < bc$, which by definition says

$$bc - ac \in F^+.$$

By the distributive law,

$$bc - ac = (b - a)c.$$

We know $b - a \in F^+$ (hypothesis) and $c \in F^+$. The first order axiom says a product of two positive elements is positive, so $(b - a)c \in F^+$, and therefore $ac < bc$.

Examples

- $F = \mathbb{Q}$, "the rational numbers," is a field — in fact an ordered field:

$$\mathbb{Q} = \left\{ \tfrac{a}{b} \;\middle|\; a, b \text{ are integers, } \; b \neq 0 \right\}, \qquad \tfrac{3}{5},\ -\tfrac{7}{2},\ 0 \in \mathbb{Q}.$$

- $\mathbb{Z} =$ the set of integers $= \mathbb{N} \cup -\mathbb{N}$ (not a field).
- $\mathbb{R}$ is an ordered field.
- $\mathbb{C} =$ the complex numbers is a field, but not an ordered one.

A non-example

Suppose F is a field in which

$$P + P = 0 \qquad \text{for every } P \in F.$$

Then F cannot be ordered. Indeed, if $P \in F^+$, then

$$P + P \in F^+,$$

but $P + P = 0$ and $0 \notin F^+$ — a contradiction.

So no P is positive, and by axiom (2) every P equals 0. That is not a field, since $0 \neq 1$.

What is still missing?

Vaguely: the ingredient we have not yet captured is

"continuity"

This is why the real numbers are called *the continuum*.

$\mathbb{Q}$ satisfies every axiom so far, and yet it has holes. We need one more axiom to say so.

Upper bounds

Definition

Let F be an ordered field and $S \subseteq F$ any subset. If $b \in F$ satisfies

$$a \leq b \qquad \text{for every } a \in S,$$

we say that b is an upper bound for S, and that S is bounded above.

Least upper bounds

Definition

Let $S \subseteq F$ be a subset. We say that $b \in F$ is a least upper bound, or a supremum, for S if

- b is an upper bound for S, and
- whenever $c \in F$ is also an upper bound for S, we have $b \leq c$.

A supremum need not belong to S, and — as we are about to see — it need not exist at all.

Complete ordered fields

Definition

F is a complete ordered field if it is an ordered field and every nonempty $S \subseteq F$ which is bounded above has a supremum.

This is the last axiom. The real numbers $\mathbb{R}$ are a complete ordered field.

An example inside $\mathbb{Q}$

Let

$$S = \left\{x \in \mathbb{Q} \mid x^2 < 9\right\} \subseteq \mathbb{Q}.$$

Claim. If $x \in S$, then $x \leq 3$.

Assume by way of contradiction that $x > 3$. Multiplying by $3 > 0$,

$$3x > 9,$$

and multiplying $x > 3$ by $x > 0$,

$$x^2 > 3x.$$

Hence $x^2 > 9$, contrary to $x \in S$. In fact 3 is a supremum for S.

$\mathbb{Q}$ is not complete

Now let

$$S = \left\{x \in \mathbb{Q} \mid x^2 < 2\right\} \subseteq \mathbb{Q}.$$

The same style of argument shows that S is bounded above: if $x \in S$, then $x < 10$. And S is nonempty, since $0 \in S$.

But S has no supremum in $\mathbb{Q}$. The only candidate would be a rational number whose square is 2 — and there is none.

$\sqrt{2} \notin \mathbb{Q}$

That is: there are no integers a, b with $b \neq 0$ such that

$$\left(\frac{a}{b}\right)^2 = 2 = \frac{2}{1}.$$

Why? If $r = \dfrac{a}{b} \in \mathbb{Q}$, we may assume a and b are not both even (otherwise cancel factors of 2). Then

$$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} = 2 \qquad \Longrightarrow \qquad a^2 = \underbrace{2b^2}_{\text{even}}.$$

$\sqrt{2} \notin \mathbb{Q}$, continued

So a^2 is even, and therefore a cannot be odd:

$$\boxed{a \text{ is even}}, \qquad \text{say } a = 2a'.$$

Substituting,

$$(2a')^2 = 2b^2, \qquad 4(a')^2 = 2b^2, \qquad 2(a')^2 = b^2.$$

The left side is even, so b is even as well. But a and b were not both even — contradiction.

Existence of square roots

Theorem (I.35)

Any nonnegative real number has a unique nonnegative square root.

Proof. Let $a \in \mathbb{R}$ with $a \geq 0$.
If $a = 0$, then $0^2 = 0 = a$. ✓

So assume $a > 0$, and set

$$S = \left\{ x \in \mathbb{R} \mid x^2 \leq a \right\}.$$

The plan

Claim. S is bounded above. Granting this, let

$$b = \sup(S).$$

Then exactly one of the following holds:

- $b^2 = a$ — what we want;
- $b^2 > a$ — we will show b is then not *least*;
- $b^2 < a$ — we will show b is then not an upper bound at all.

Ruling out the last two leaves $b^2 = a$.

An observation

Observe

If $a > 0$, then $(1 + a)^2 > a$.

Why? Each of a, 1, a^2 is positive, so

$$1 + a + a^2 > 0.$$

Adding a to both sides,

$$1 + 2a + a^2 > a, \qquad \text{that is,} \qquad (1 + a)^2 > a.$$

$1 + a$ is an upper bound for S

Claim. If $x^2 \leq a$, then $x \leq 1 + a$.

If $x < 0$ this is clear, so assume $x > 0$ and argue by contradiction: suppose $x > 1 + a$. Multiplying by $x > 0$,

$$x^2 > x(1 + a),$$

and multiplying $x > 1 + a$ by $1 + a > 0$,

$$x(1 + a) > (1 + a)^2.$$

Hence $x^2 > (1 + a)^2 > a$, a contradiction. So S is bounded above, and $b = \sup(S)$ exists.

Case $b^2 > a$: b is not least

We look for a c with $0 < c < b$ and $c^2 > a$. Take

$$c = b - \frac{b^2 - a}{2b}.$$

Claim

$0 < c < b$.

Both halves reduce to a comparison we can multiply through by $2b > 0$.

Proof of the claim

Why $c < b$. Subtracting b, the inequality $c < b$ says exactly

$$\frac{b^2 - a}{2b} = b - c > 0.$$

Since $2b > 0$, this holds if and only if $b^2 - a > 0$, that is, if and only if $b^2 > a$, which is exactly the assumption of this case.

Why $c > 0$. The inequality $c > 0$ together with $c = b - \frac{b^2-a}{2b}$ says exactly

$$b > \frac{b^2 - a}{2b}.$$

Multiplying by $2b > 0$, this holds if and only if $2b^2 > b^2 - a$, that is, if and only if $b^2 > -a$. And $b^2 > 0 > -a$, since $a > 0$.

We have now established $0 < c < b$, where $c = b - \frac{b^2 - a}{2b}$.
Squaring this gives us:

$$c^2 = b^2 - (b^2 - a) + \frac{(b^2 - a)^2}{(2b)^2} = a + \underbrace{\frac{(b^2 - a)^2}{(2b)^2}}_{\text{positive}} > a.$$

So for every $x \in S$ we have

$$c^2 > a \geq x^2, \qquad \text{hence} \qquad c > x.^{1}$$

That makes c an upper bound for S with $c < b$, contradicting the fact that b is the least upper bound.

So $b^2 > a$ is impossible.

[1] for this you need to check that if s and t are positive then $s > t$ if and only if $s^2 > t^2$

Case $b^2 < a$: b is not an upper bound

On the other hand, suppose $b^2 < a$. The numerator $a - b^2$ is then positive by assumption, and $3b > 0$, so

$$\frac{a - b^2}{3b} > 0.$$

Both b and this quantity are positive, so we may choose a c that is positive and smaller than each of them:

$$c > 0, \qquad c < b, \qquad c < \frac{a - b^2}{3b}.$$

We want to show $b + c \in S$, that is, $(b + c)^2 \leq a$. Since $c > 0$ this gives

$$b < b + c \in S,$$

so b is not an upper bound for S after all.

Case $b^2 < a$, continued

Expanding and collecting the terms containing c,

$$(b+c)^2 = b^2 + 2bc + c^2 = b^2 + c(2b+c).$$

Now use $c < b$: adding $2b$ to both sides gives $2b + c < 2b + b = 3b$. Multiplying that by $c > 0$ gives $c(2b+c) < 3bc$, and hence

$$(b+c)^2 < b^2 + 3bc.$$

Finally $c < \dfrac{a-b^2}{3b}$, and multiplying by $3b > 0$ gives $3bc < a - b^2$. Therefore

$$(b+c)^2 < b^2 + (a-b^2) = a. \qquad \checkmark$$

So $b^2 < a$ is impossible as well, and therefore $b^2 = a$. $\square$