Honors Calculus 1610

Lecture 4 (extended): Dedekind Cuts, Complex Numbers, and the Problem of Area

September 11, 2026

Today

Announcement

I will not be here on Monday.

- Finish the real numbers (Dedekind cuts).
- A digression into other number systems (complex, and “hypercomplex”).
- Start heading towards integrals.

Notation

$\exists$	there exists
$\forall$	for all
st	such that
$\square$	that's the end of the proof

An assumption

Principle

If S is a finite, nonempty subset of $\mathbb{R}$, then S has a largest element and a smallest element.

We take this as given. It is essentially axiomatic — part of what we mean by *finite*.

The floor of a real number

Proposition

If $a \in \mathbb{R}$, then there is an integer $\lfloor a \rfloor \in \mathbb{Z}$ such that

$$\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1.$$

Every real number lies in a half-open interval $[m, m+1)$ with integer endpoints, and $\lfloor a \rfloor = m$ is the integer at the left end.

We prove it first for $a > 0$, and then reduce the general case to that one.

Proof, when $a \in \mathbb{R}^+$

Consider

$$S = \{\, n \in \mathbb{N} \;:\; n \leq a \,\}\,.$$

- S is nonempty: since $a > 0$ we have $0 \in S$.
- S is bounded above by a, by definition of S.

Why is S finite?

Why S is finite

Because S is bounded, and

the natural numbers have no upper bound

(this is the recitation problem).

So there is some $m \in \mathbb{N}$ with $m > a$, and therefore $m \notin S$. Every element of S is a natural number smaller than m, and there are only finitely many of those. Hence

$$S \text{ is finite.}$$

Proof, concluded

S is finite and nonempty, so it has a maximum element; call it n.

Since $n \in S$ we have $n \leq a$. Since n is the largest element of S, we have $n + 1 \notin S$, that is,

$$n + 1 > a.$$

Together,

$$n \leq a < n + 1,$$

so setting $\lfloor a \rfloor := n$ finishes the positive case. $\square$

Aside: what if a is not positive?

Suppose $-a \in \mathbb{R}^+$. Unboundedness of $\mathbb{N}$ gives some $m \in \mathbb{N}$ with

$$m > -a, \qquad \text{so} \qquad m + a > 0.$$

The positive case applies to $m + a$, and

$$\lfloor m + a \rfloor = m + \lfloor a \rfloor ,$$

in other words

$$\boxed{\lfloor a \rfloor = \lfloor m + a \rfloor - m}$$

Subtracting m from $\lfloor m + a \rfloor \leq m + a < \lfloor m + a \rfloor + 1$ gives the required inequalities for a.

Getting close to zero

Proposition

If $a \in \mathbb{R}^+$, then there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < a.$$

Proof

Unboundedness of $\mathbb{N}$: there is an $n \in \mathbb{N}$ with

$$n > \frac{1}{a}.$$

Multiply by $a > 0$:

$$an > 1.$$

Multiply by $\frac{1}{n} > 0$:

$$a > \frac{1}{n}. \qquad \square$$

Density of $\mathbb{Q}$ in $\mathbb{R}$

Real numbers are conventionally described by a series of increasingly close approximations by rational numbers, which is one way to think about decimal expansions:

$$e = 2.7182818\ldots \sim 2.71828 = \frac{271828}{100000}.$$

Proposition

If $a < b$ are real numbers, then there exists $r \in \mathbb{Q}$ with

$$a < r \leq b.$$

Proof

Since $b - a > 0$, the previous proposition gives an $n \in \mathbb{N}$ with

$$\frac{1}{n} < b - a.$$

Multiplying by $n > 0$ gives $1 < bn - an$, that is,

$$an + 1 < bn.$$

Let $m = \lfloor bn \rfloor$, so that

$$m \leq bn < m + 1.$$

Claim. $r = \dfrac{m}{n}$ works.

Proof: the two inequalities

First, $r \leq b$. From $m \leq bn$, divide by $n > 0$:

$$\frac{m}{n} \leq b.$$

Second, $a < r$. Chain the two facts:

$$an + 1 < bn < m + 1.$$

So $an + 1 < m + 1$, hence $an < m$, hence

$$a < \frac{m}{n}. \qquad \square$$

What is a real number?

We defined the notion of a

“complete ordered field”

and made the claim: $\mathbb{R}$ **is the only one.**

Question: is there one?

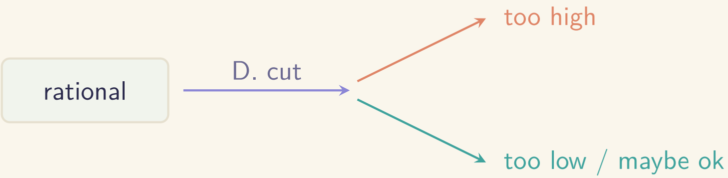
Decimals are not unique

$$0.9999\cdots = 1 = 1.000\cdots$$

So a real number is not the same thing as a decimal expansion.

Dedekind cuts, informally

A Dedekind cut is a procedure for playing a guessing game with rational numbers, in order to specify a real number.



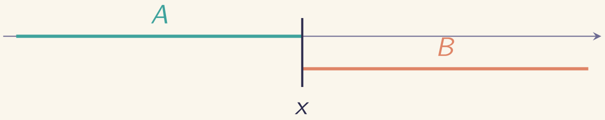
The cut attached to a real number

If $x \in \mathbb{R}$, define

$$L_x = \{\, r \in \mathbb{Q} \;:\; r \le x \,\}, \qquad U_x = \{\, r \in \mathbb{Q} \;:\; r > x \,\}.$$

By density of $\mathbb{Q}$,

$$x = \inf\left(U_x\right).$$



Dedekind cuts, formally

Definition

A Dedekind cut is a pair of subsets $A, B \subseteq \mathbb{Q}$ such that

- A and B are both nonempty;
- $\forall\, a \in A,\ b \in B \implies a \leq b$;
- $A \cup B = \mathbb{Q}$;
- B does not contain its infimum: $\inf(B) \notin B$.

The last condition gives exactly one cut for each real number.

A digression: other number systems

At some point it became useful to have a square root of -1.

Once you have it, you have everything else:

every polynomial factors completely over the complex numbers.

From ad hoc rules to a definition

Originally i was manipulated in an ad hoc way.

Compare fractions. The rule

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

can be derived from rules about division, or taken as the definition of addition of fractions.

The complex numbers, axiomatically

A complex number is a pair of real numbers (a, b), with

$$(a, b) + (c, d) = (a + c,\ b + d),$$

$$(a, b) \cdot (c, d) = (ac - bd,\ ad + bc).$$

Writing $(a, b) = a + bi$ and using $i^2 = -1$:

$$(a + bi)(c + di) = ac + adi + bci + bd\, i^2 = (ac - bd) + (ad + bc)i.$$

One checks that this gives a field, which we write $\mathbb{C}$.

$\mathbb{C}$ cannot be ordered

In an ordered field every square satisfies $a^2 \geq 0$. So in any ordering of $\mathbb{C}$:

- $1 = 1^2$ is positive, hence -1 is negative;
- but $-1 = i^2$ is a square, hence $-1 \geq 0$.

Contradiction. So $\mathbb{C}$ admits no ordering.

Sidebar: why did this matter?

- A unified context for solutions of arbitrary polynomial equations.
- A revolution in plane geometry, Euclidean and then non-Euclidean: multiplication by a complex number is a rotation and scaling of the plane.

Hamilton: what about three dimensions?

The plane works. What about space?

After years of turmoil and tribulation, Hamilton found the answer lives in four dimensions: the quaternions $\mathbb{H}$, and the source of the i, j, k notation still used for vectors.

$$i^2 = j^2 = k^2 = ijk = -1.$$

Multiplication is not commutative: $ij = -ji$.

Broome Bridge, 16 October 1843

It came to him walking the Royal Canal, and he cut it into the bridge:

> Here as he walked by
> on the 16th of October 1843
> Sir William Rowan Hamilton
> in a flash of genius discovered
> the fundamental formula for
> quaternion multiplication
> $i^2 = j^2 = k^2 = ijk = -1$
> & cut it on a stone of this bridge.

The *Hamilton Walk* from Dunsink Observatory to Broom Bridge is held every 16 October, and an Irish postage stamp carries the formula too.

And then it stops

A few years later: the octonions, in eight dimensions. Multiplication there is not even associative.

They encode rotations in 7 and 8 dimensions — another story entirely.

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

Back to calculus

We will soon start integration. First: functions and areas.

Ordered pairs

Definition

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Exercise

Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Cases: $a = b$, and $a \neq b$.

Cartesian products, relations, functions

- The Cartesian product $X \times Y$ is the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$.
- A relation from X to Y is a subset $R \subseteq X \times Y$.
- A function $f : X \to Y$ is a relation in which every $x \in X$ appears in exactly one pair.

A function is its graph; the domain and codomain are part of the data.

Practice

For $f\colon X \to Y$:

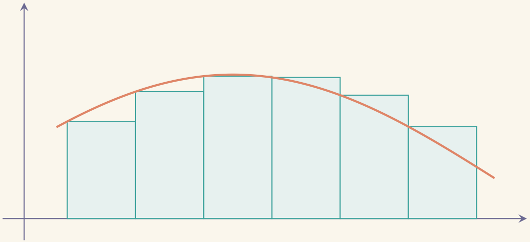
- f is injective if $f(a) = f(b)$ forces $a = b$;
- f is surjective if every $y \in Y$ is $f(x)$ for some x;
- f is a bijection if it is both.

To check

f is a bijection if and only if it is invertible: there is a $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

The intuitive integral

The integral is the area under a graph; the area of a rectangle is base times height.



Fill the region with rectangles, add the areas, refine.

Is a segment the sum of its points?

$[0,1]$ is the union of the points it contains, and each point has length 0. If lengths add, then

$$1 = \sum_{x\in[0,1]} 0 = 0.$$

There is no sum of uncountably many numbers. So additivity cannot hold for arbitrary unions.

Salt and pepper

Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Every subinterval of $[0,1]$ contains both rationals and irrationals.

What is $\int_0^1 f$? Is it 0? Is it 1?

Banach–Tarski

Let $B \subseteq \mathbb{R}^3$ be the standard unit ball. There are subsets

$$S_1, \ldots, S_5 \subseteq B$$

which, moved only by rotations and translations, reassemble into two balls: B_1 from S_1, S_2 and B_2 from S_3, S_4, S_5, each a translate of B.

The pieces have no volume.

R. M. Robinson, *On the decomposition of spheres*, Fund. Math. **34** (1947), 246–260.

What survives

every set has an area	no — Banach–Tarski
areas add over arbitrary unions	no — points in $[0, 1]$
congruent sets have equal area	keep
rectangle $=$ base $\times$ height	keep

Next: axioms for the area of subsets of the plane.