Honors Calculus 1610

Lecture 4: Floors, Density, and Dedekind Cuts

September 11, 2026

Today

Announcement

I will not be here on Monday.

- Finish the real numbers (Dedekind cuts).
- A digression into other number systems (complex, and "hypercomplex").
- Start heading towards integrals.

Notation

$\exists$	there exists
$\forall$	for all
st	such that
$\square$	that's the end of the proof

An assumption

Principle

If S is a finite, nonempty subset of $\mathbb{R}$, then S has a largest element and a smallest element.

We take this as given. It is essentially axiomatic — part of what we mean by *finite*.

The floor of a real number

Proposition

If $a \in \mathbb{R}$, then there is an integer $\lfloor a \rfloor \in \mathbb{Z}$ such that

$$\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1.$$

Every real number lies in a half-open interval $[m, m+1)$ with integer endpoints, and $\lfloor a \rfloor = m$ is the integer at the left end.

We prove it first for $a > 0$, and then reduce the general case to that one.

Proof, when $a \in \mathbb{R}^+$

Consider

$$S = \{\, n \in \mathbb{N} \;:\; n \leq a \,\}\,.$$

- S is nonempty: since $a > 0$ we have $0 \in S$.
- S is bounded above by a, by definition of S.

Why is S finite?

Why S is finite

Because S is bounded, and

the natural numbers have no upper bound

(this is the recitation problem).

So there is some $m \in \mathbb{N}$ with $m > a$, and therefore $m \notin S$. Every element of S is a natural number smaller than m, and there are only finitely many of those. Hence

S is finite.

Proof, concluded

S is finite and nonempty, so it has a maximum element; call it n.

Since $n \in S$ we have $n \leq a$. Since n is the largest element of S, we have $n+1 \notin S$, that is,

$$n+1 > a.$$

Together,

$$n \leq a < n+1,$$

so setting $\lfloor a \rfloor := n$ finishes the positive case. $\square$

Aside: what if a is not positive?

Suppose $-a \in \mathbb{R}^+$. Unboundedness of $\mathbb{N}$ gives some $m \in \mathbb{N}$ with

$$m > -a, \qquad \text{so} \qquad m + a > 0.$$

The positive case applies to $m + a$, and

$$\lfloor m + a \rfloor = m + \lfloor a \rfloor ,$$

in other words

$$\boxed{\lfloor a \rfloor = \lfloor m + a \rfloor - m}$$

Subtracting m from $\lfloor m + a \rfloor \leq m + a < \lfloor m + a \rfloor + 1$ gives the required inequalities for a.

Getting close to zero

Proposition

If $a \in \mathbb{R}^+$, then there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < a.$$

Proof

Unboundedness of $\mathbb{N}$: there is an $n \in \mathbb{N}$ with

$$n > \frac{1}{a}.$$

Multiply by $a > 0$:

$$an > 1.$$

Multiply by $\frac{1}{n} > 0$:

$$a > \frac{1}{n}. \qquad \square$$

Density of $\mathbb{Q}$ in $\mathbb{R}$

Real numbers are conventionally described by a series of increasingly close approximations by rational numbers, which is one way to think about decimal expansions:

$$e = 2.7182818\ldots \sim 2.71828 = \frac{271828}{100000}.$$

Proposition

If $a < b$ are real numbers, then there exists $r \in \mathbb{Q}$ with

$$a < r \leq b.$$

Proof

Since $b - a > 0$, the previous proposition gives an $n \in \mathbb{N}$ with

$$\frac{1}{n} < b - a.$$

Multiplying by $n > 0$ gives $1 < bn - an$, that is,

$$an + 1 < bn.$$

Let $m = \lfloor bn \rfloor$, so that

$$m \leq bn < m + 1.$$

Claim. $r = \dfrac{m}{n}$ works.

Proof: the two inequalities

First, $r \le b$. From $m \le bn$, divide by $n > 0$:

$$\frac{m}{n} \le b.$$

Second, $a < r$. Chain the two facts:

$$an + 1 < bn < m + 1.$$

So $an + 1 < m + 1$, hence $an < m$, hence

$$a < \frac{m}{n}. \qquad \square$$

What is a real number?

We defined the notion of a

“complete ordered field”

and made the claim: $\mathbb{R}$ **is the only one.**

Question: is there one?

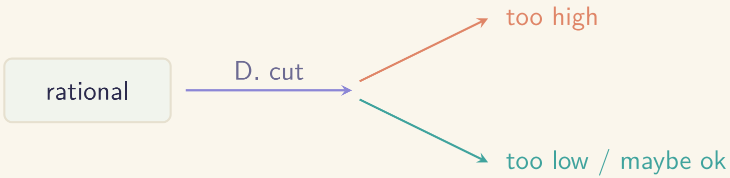
Decimals are not unique

$$0.9999\cdots = 1 = 1.000\cdots$$

So a real number is not the same thing as a decimal expansion.

Dedekind cuts, informally

A Dedekind cut is a procedure for playing a guessing game with rational numbers, in order to specify a real number.



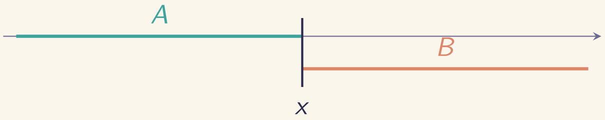
The cut attached to a real number

If $x \in \mathbb{R}$, define

$$L_x = \{\, r \in \mathbb{Q} \;:\; r \le x \,\}, \qquad U_x = \{\, r \in \mathbb{Q} \;:\; r > x \,\}.$$

By density of $\mathbb{Q}$,

$$x = \inf\left(U_x\right).$$



Dedekind cuts, formally

Definition

A Dedekind cut is a pair of subsets $A, B \subseteq \mathbb{Q}$ such that

- A and B are both nonempty;
- $\forall\, a \in A,\ b \in B \implies a \leq b$;
- $A \cup B = \mathbb{Q}$;
- B does not contain its infimum: $\inf(B) \notin B$.

The last condition gives exactly one cut for each real number.

Back to calculus

We will soon start integration. First: functions and areas.

Ordered pairs

Definition

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Exercise

Show that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Cases: $a = b$, and $a \neq b$.

Cartesian products, relations, functions

- The Cartesian product $X \times Y$ is the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$.
- A relation from X to Y is a subset $R \subseteq X \times Y$.
- A function $f\colon X \to Y$ is a relation in which every $x \in X$ appears in exactly one pair.

A function is its graph; the domain and codomain are part of the data.