Honors Calculus 1610

Lecture 5: Area, Step Functions, and the Integral

September 18, 2026

Where we are

Course outline

- Chunk 1: precalculus
- Chunk 2: calculus (1 variable), plus a bit of vectors
- Chunk 3: multivariable

Last time

- Finished the real numbers (Dedekind cuts).
- Cartesian products, ordered pairs, functions.

Before the exam

- (Definite) integrals.
- Limits and continuity.
- Intermediate value theorem, etc.

Story time: the complex and hypercomplex numbers

First: a square root of -1

At some point it became useful to have a square root of -1:

$$i^2 = -1.$$

The standard representation of the resulting system is $\mathbb{R}$ together with i, with

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Axiomatizing $\mathbb{C}$

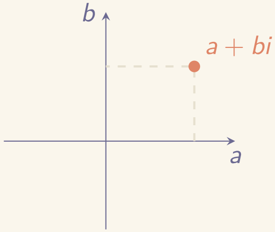
A complex number is a pair of real numbers (a, b), with

$$(a, b) + (c, d) = (a + c,\ b + d),$$

$$(a, b) \cdot (c, d) = (ac - bd,\ ad + bc),$$

and the identification

$$(a, b) \longleftrightarrow a + bi.$$



Two dimensions, then three?

The complex numbers were a revolution in two-dimensional geometry in the 1700s. What about three dimensions?

Hamilton found that the answer lives in four dimensions: the quaternions $\mathbb{H}$, with i, j, k and

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik.$$

Multiplication is not commutative.

Quaternion arithmetic, in full

$$(a + bi + cj + dk) + (e + fi + gj + hk) = (a + e) + (b + f)i + (c + g)j + (d + h)k,$$

$$\begin{aligned}(a + bi + cj + dk)(e + fi + gj + hk) \\ &= (ae - bf - cg - dh) \\ &+ (af + be + ch - dg)\,i \\ &+ (ag - bh + ce + df)\,j \\ &+ (ah + bg - cf + de)\,k.\end{aligned}$$

Quaternions and rotations

Write a vector in three-space as

$$v = ai + bj + ck,$$

and let $q = d + ei + fj + gk$ be a quaternion. Then

$$q\, v\, q^{-1} \;=\; \text{“}v \text{ rotated by } q\text{”}.$$

This is where the i, j, k notation for vectors comes from.

And then it stops: the octonions

A few years later came the octonions $\mathbb{O}$, in eight dimensions: seven imaginary units $e_1, \ldots, e_7$, each with $e_n^2 = -1$, multiplied according to a fixed rule.

Multiplication there is not even associative.

They are related to rotations in 7 and 8 dimensions — 8 in particular.

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

Back to (pre)calculus

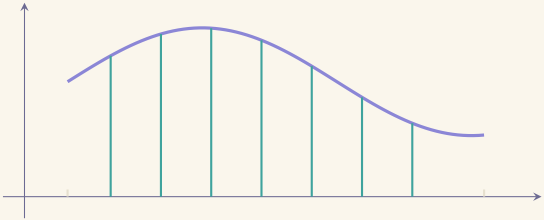
Warm up: which of these are functions?

$$
\begin{aligned}
&1) \quad \left\{(x,y) \in \mathbb{Z} \times \mathbb{R} \;\middle|\; y = x^2\right\} \\
&2) \quad \left\{(x,y) \in \mathbb{R} \times \mathbb{Z} \;\middle|\; y = x^2\right\} \\
&3) \quad \left\{(x,y) \in \mathbb{R} \times \mathbb{R} \;\middle|\; y = x^2\right\} \\
&4) \quad \left\{(x,y) \in \mathbb{R} \times \mathbb{R} \;\middle|\; y^2 = x\right\} \\
&5) \quad \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is even}\}
\end{aligned}
$$

Recall: $f : X \to Y$ is a subset $f \subseteq X \times Y$ such that for every x there is exactly one y with $(x,y) \in f$.

Area (§1.6)

The integral we are after is the area under a graph.



But what *is* area? Before defining it we should be warned that "size" can behave very badly.

A first warning

$$\mathbb{Z} = \{\text{even integers}\} \cup \{\text{odd integers}\},$$

yet

$$\#\{\text{even integers}\} = \#\mathbb{Z}.$$

A set can be split into pieces each of which is “as big as” the whole.

Banach–Tarski

Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

One can find subsets

$$A_1, A_2, B_1, B_2, B_3 \subseteq S,$$

all mutually disjoint, whose union is S, and rigid rotations and translations $A_i \rightsquigarrow A_i'$, $B_j \rightsquigarrow B_j'$, such that

$$A_1' \cup A_2' = S, \qquad B_1' \cup B_2' \cup B_3' = S.$$

So the pieces cannot all have a well-defined area.

The concept: measurability

We are not going to say, once and for all, which sets have an area. What we write down instead is an axiomatic framework for what it means to *have* a concept of measurability: a special collection of subsets, whose members we call measurable, together with an area function on them.

$$\mathcal{M} = \text{some set of subsets of } \mathbb{R}^2, \qquad a\colon \mathcal{M} \longrightarrow \mathbb{R}^+,$$

so $a(S) \in \mathbb{R}^+$ for S measurable.

The axioms below do not by themselves pick out a single such concept.

The axioms

- If S, T are measurable, so are $S \cup T$, $S \cap T$, $S \setminus T$, and

$$a(S \cup T) = a(S) + a(T) - a(S \cap T).$$

- $a(\emptyset) = 0$.
- a is invariant under translation.
- The unit square has area 1.
- Exhaustion.

etc.

From these one *deduces* that a rectangle with sides a and b has area ab — it is not an axiom.

Exhaustion

Axiom

Let $T \subseteq \mathbb{R}^2$ and set

$$A = \{a(S) \mid S \subseteq T,\ S \text{ measurable}\}, \qquad B = \{a(S) \mid T \subseteq S,\ S \text{ measurable}\}.$$

If $\sup(A) = \inf(B) = L$, then T is measurable and $a(T) = L$.

This is the moral of Dedekind cuts again: we pin a real number down by what lies above it and what lies below it, and if the two agree, we have our number.

Step functions

Definition

$f : [a, b] \to \mathbb{R}$ is a step function if there are

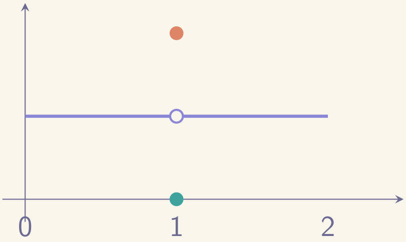
$$x_0, x_1, \ldots, x_n \in [a, b], \qquad x_0 = a, \quad x_n = b, \quad x_i < x_{i+1},$$

with $f(x) = c_i$ constant for $x \in (x_{i-1}, x_i)$.

Note the open intervals: the values at the subdivision points are not mentioned, and will not matter.

The values at the jumps do not matter

Take $f(x) = 1$ for $x \in [0,2] \setminus \{1\}$, and let $f(1)$ be whatever you like — say 2, or 0.



A single point contributes no area, and the answer is 2 either way.

The integral of a step function

Definition

If f is a step function as above,

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} c_i\,(x_i - x_{i-1}).$$

When $f \geq 0$ this is exactly the area of the rectangles under the graph, so it agrees with the area function of the previous slides.

Properties of the integral of a step function

- Linearity: $\int_a^b (f+g) = \int_a^b f + \int_a^b g$, and $\int_a^b \lambda f = \lambda \int_a^b f$.
- Monotonicity: if $f(x) \le g(x)$ for all x, then $\int_a^b f \le \int_a^b g$.
- Additivity over intervals: $\int_a^b f + \int_b^c f = \int_a^c f$.
- $\int_a^a f = 0$, so it is consistent to define $\int_b^a f = -\int_a^b f$ and have the integral defined for any ordering of the endpoints.

Integrability

Let $f\colon [a,b] \to \mathbb{R}$ be a function, and set

$$L = \sup\left\{ \int_a^b s(x)\,dx \;\middle|\; s \text{ a step function, } s \le f \right\},$$

$$U = \inf\left\{ \int_a^b t(x)\,dx \;\middle|\; t \text{ a step function, } f \le t \right\},$$

where $s \le f$ means $s(x) \le f(x)$ for all x.

Definition

If $U = L$ we say f is integrable, and define $\int_a^b f(x)\,dx = L = U$.

Aside

If $f(x) \geq 0$ always, then $\int_a^b f(x)\,dx$, if it exists, is the area under the curve — in the sense of the exhaustion axiom, with the step functions below and above playing the roles of A and B.

These all hold for arbitrary integrable functions

Suppose f and g are integrable, with

$$s_1 \le f \le t_1, \qquad s_2 \le g \le t_2,$$

step functions, and write $L_1 = \int_a^b f$, $L_2 = \int_a^b g$.

Then $s_1 + s_2 \le f + g \le t_1 + t_2$, and

$$\int_a^b s_1 \le L_1, \qquad \int_a^b s_2 \le L_2.$$

Additivity, concluded

Given any $\varepsilon > 0$, choose step functions s_1, s_2 with

$$\int_a^b s_1 > L_1 - \frac{\varepsilon}{2}, \qquad \int_a^b s_2 > L_2 - \frac{\varepsilon}{2}.$$

Then $s_1 + s_2$ is a step function below $f + g$, and

$$\int_a^b (s_1 + s_2) = \int_a^b s_1 + \int_a^b s_2 > L_1 + L_2 - \varepsilon.$$

Since ε was arbitrary, the lower integral of $f + g$ is $L_1 + L_2$; the same argument from above finishes it. $\square$

Integrating a power

Theorem (1.15)

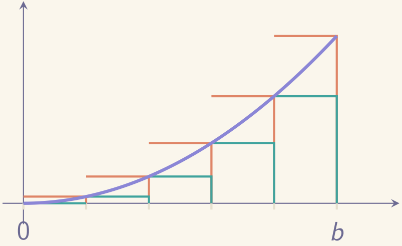
If p is a positive integer and $b > 0$, then

$$\int_0^b x^p \, dx = \frac{b^{p+1}}{p+1}.$$

The inequality behind it

Cut $[0, b]$ into n equal pieces and compare x^p with the step functions below and above it. Everything comes down to

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p.$$



Proof of the inequality, by induction

Write $m = n - 2$, and let $P(m)$ be the statement

$$\sum_{k=1}^{m+1} k^p < \frac{(m+2)^{p+1}}{p+1}.$$

Base case $P(0)$**:**

$$1 < \frac{2^{p+1}}{p+1}, \qquad \text{that is,} \qquad p+1 < 2^{p+1}.$$

A side induction

Let $Q(p)$ be the statement $p+1 < 2^{p+1}$.

$Q(0)$: $\quad 1 < 2.\ \checkmark$

$Q(p+1)$:

$$(p+1)+1 \;<\; 2^{p+1}\cdot 2 \;=\; 2^{p+1}+2^{p+1},$$

which holds since $p+1 < 2^{p+1}$ and $1 < 2^{p+1}$. $\square$

The inductive step

Assume $P(m)$. Then

$$\sum_{k=1}^{m+2} k^p = \sum_{k=1}^{m+1} k^p + (m+2)^p < \frac{(m+2)^{p+1}}{p+1} + (m+2)^p.$$

On the other hand, expanding the binomial,

$$\begin{aligned}\frac{\big((m+2)+1\big)^{p+1}}{p+1} &= \frac{(m+2)^{p+1}}{p+1} + \frac{(p+1)(m+2)^p}{p+1} + \cdots \\ &= \frac{(m+2)^{p+1}}{p+1} + (m+2)^p + \cdots\end{aligned}$$

and every remaining term is positive. This gives $P(m+1)$. $\square$

Next time

Limits and continuity.