

MATH 1610 / HONORS CALCULUS

Honors Calculus 1610

Lecture 8: The Intermediate Value Theorem and Friends

October 9, 2026

Today

- ◆ Finish integral calculus.
- ◆ Make some progress with differential calculus.

Intermediate value theorem & friends

Last time: Bolzano's theorem

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, with the signs of $f(a)$ and $f(b)$ **different**, then there is $c \in (a, b)$ with $f(c) = 0$.

The proof: let $S = \{x \in [a, b] \mid f(x) < 0\}$ and take $c = \sup(S)$. The lemma on the **next slide** is what rules out $f(c) < 0$ and $f(c) > 0$.

The sign preservation lemma

This is the lemma we proved last time; it is worth giving it a name.

Lemma (sign preservation)

If f is continuous at c and $f(c) \neq 0$, then there is an $h > 0$ such that $f(x)$ has the **same sign** as $f(c)$ for every $x \in (c - h, c + h)$.

The proof was one line: apply continuity at c with

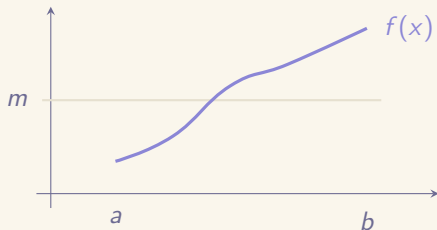
$$\epsilon = |f(c)|,$$

and let h be the resulting δ . Then $f(x)$ lies within $|f(c)|$ of $f(c)$, which is not far enough to change its sign.

The intermediate value theorem

Theorem

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a) < m < f(b)$, then there exists $c \in (a, b)$ with $f(c) = m$.



Proof

Define $g(x) = f(x) - m$, which is continuous. Then

$$g(a) = f(a) - m < 0, \quad g(b) = f(b) - m > 0.$$

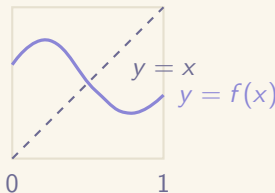
By Bolzano's theorem there is $c \in (a, b)$ with $g(c) = 0$. But

$$0 = g(c) = f(c) - m \quad \implies \quad f(c) = m. \quad \square$$

An application: Brouwer's fixed point theorem

Theorem

If $f: [0, 1] \rightarrow [0, 1]$ is continuous, then $f(c) = c$ for some $c \in [0, 1]$.



idea: The graph lives in the square, and has to cross the diagonal.

Proof

If $f(0) = 0$ we are done. Otherwise, since f takes values in $[0, 1]$, we have $f(0) > 0$.

If $f(1) = 1$ we are done. Otherwise $f(1) < 1$.

So consider

$$g(x) = f(x) - x,$$

which is continuous. Then

$$g(0) = f(0) > 0, \quad g(1) = f(1) - 1 < 0.$$

By Bolzano's theorem, $g(c) = 0$ for some $c \in (0, 1)$. But $0 = g(c) = f(c) - c$, so $f(c) = c$. $\square$

An application: n -th roots

Recall that $f(x) = x^n$, for n positive, is continuous — it is a polynomial.

Let a be a positive real number, and choose

$$c > \max \{1, a\}.$$

Then $0 < a \leq c \leq c^n$, so

$$f(0) = 0, \quad f(c) = c^n \geq a.$$

Why $c \leq c^n$

Since $1 \leq c$, multiplying through by c gives $c \leq c^2$.

Multiplying $c \leq c^2$ by c again gives $c^2 \leq c^3$, and then $c^3 \leq c^4$, and so on:

$$c \leq c^2 \leq c^3 \leq c^4 \leq \dots \leq c^n.$$

(An induction, with the ordered field axioms doing the work at each step.)

n -th roots exist

If $a = c^n$, then c is an n -th root of a and we are done.

Otherwise $a < c^n$, and then

$$0 = f(0) < a < c^n = f(c).$$

So by the intermediate value theorem there is $b \in (0, c)$ with

$$f(b) = a, \quad \text{that is,} \quad b^n = a.$$

Every positive real number has a positive real n -th root.

Is it unique?

For that we need to know that $x \mapsto x^n$ cannot take the same value twice. So we need some vocabulary.

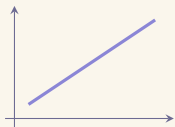
Definition

If $S \subseteq \mathbb{R}$ and $f: S \rightarrow \mathbb{R}$, we say f is

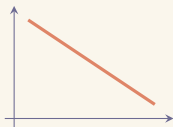
- ♦ (strictly) increasing if $f(b) > f(a)$ whenever $b > a$ in S ;
- ♦ nondecreasing if $f(b) \geq f(a)$ whenever $b > a$;
- ♦ (strictly) decreasing if $f(b) < f(a)$ whenever $b > a$;
- ♦ nonincreasing if $f(b) \leq f(a)$ whenever $b > a$.

Strictly increasing or strictly decreasing: **strictly monotone**.

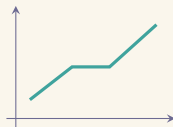
The four pictures



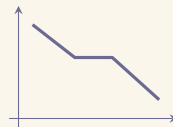
increasing



decreasing



nondecreasing



nonincreasing

The two on the **left** are the strict ones; the two on the right are allowed to go flat.

Recall: injective and surjective

Let $f: X \rightarrow Y$ be a function between two sets. We say f is

- ♦ **one to one** if $f(x_1) = f(x_2) \implies x_1 = x_2$ — also called *injective*, or *monic*;
- ♦ **onto** if for every $y \in Y$ there is $x \in X$ with $f(x) = y$ — also called *surjective*, or *epic*.

If f is both, we say f is **bijective**.

Bijections and inverses

Lemma

$f: X \rightarrow Y$ is bijective if and only if there is an **inverse function** $f^{-1}: Y \rightarrow X$ such that

$$f \circ f^{-1}(y) = y \quad \text{and} \quad f^{-1} \circ f(x) = x.$$

Strictly monotone $\implies$ injective

Remark

If $f: S \rightarrow \mathbb{R}$ is strictly monotonic, then f is injective.

Say f is increasing. We want $f(x_1) = f(x_2) \implies x_1 = x_2$.

Suppose instead $f(x_1) = f(x_2)$ and $x_1 \neq x_2$. Then either $x_1 < x_2$ or $x_2 < x_1$; say $x_1 < x_2$. Since f is increasing,

$$f(x_1) < f(x_2),$$

contradicting $f(x_1) = f(x_2)$. $\square$

x^n is monotonic on $\mathbb{R}^+$

Suppose $0 < a < b$. Multiplying $a < b$ by a and by b :

$$a^2 < ab, \quad ab < b^2, \quad \text{so} \quad a^2 < b^2.$$

Multiplying again, $a^2b < ab^2$ and $a^3 < ab^2$ and $ab^2 < b^3$, giving $a^3 < b^3$; and so on.

The general step

If $A, B, C, D > 0$ with $A < B$ and $C < D$, then multiplying the first by C and the second by B ,

$$AC < BC \quad \text{and} \quad BC < BD, \quad \text{so} \quad AC < BD.$$

So the root is unique

$x \mapsto x^n$ is strictly increasing on $(0, \infty)$, hence **injective** there.

So there is exactly one positive b with $b^n = a$:

every positive real number has a **unique** positive n -th root.

Equivalently: $x \mapsto x^n$ is a bijection $(0, \infty) \rightarrow (0, \infty)$, and its inverse is $x \mapsto \sqrt[n]{x}$.

Boundedness

Theorem

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f is **bounded**.

Suppose not. Bisect $[a, b]$: f must be unbounded on at least one half. Subdivide that “unbounded” half over and over.

Let c_i be the left endpoint of the i -th subdivision. We get a sequence of points

$$c_1 \leq c_2 \leq \cdots \leq c_n \leq \cdots,$$

and we set

$$c = \sup \{c_1, c_2, \dots\}, \quad r = f(c).$$

The contradiction

Let $\epsilon = 1 > 0$. By continuity at c there is $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - r| < \epsilon = 1,$$

and hence $f(x) < r + 1$ for every $x \in (c - \delta, c + \delta)$.

Now choose n large enough that the width of the n -th subdivision, which is $\frac{1}{2^n}$, satisfies $\frac{1}{2^n} < \delta$. Then:

- ♦ f is **unbounded** on that small interval near c ;
- ♦ f is **bounded** on $(c - \delta, c + \delta)$, which contains it.

Contradiction. $\square$

Conclusion so far

$f: [a, b] \rightarrow \mathbb{R}$ continuous $\implies f$ is bounded,

so there are m, M with

$$m \leq f(x) \leq M \quad \text{for all } x \in [a, b].$$

To show the bounds are **reached**, take the best ones:

$$m = \inf \{f(x)\}, \quad M = \sup \{f(x)\}.$$

The extreme value theorem

Theorem

A continuous $f: [a, b] \rightarrow \mathbb{R}$ attains its infimum and its supremum.

Suppose $f(x) = M$ has no solutions. Then

$$g(x) = \frac{1}{M - f(x)}$$

has a nonvanishing denominator, so it is continuous on $[a, b]$ — and therefore bounded, by the previous theorem. Say $g(x) < R$ for all x .

Finishing the extreme value theorem

From $\frac{1}{M - f(x)} < R$, and noting $M - f(x) > 0$:

$$1 < R(M - f(x)) \implies 1 < RM - Rf(x) \implies f(x) < M - \frac{1}{R}.$$

So $M - \frac{1}{R}$ is an upper bound for f which is **smaller** than M — so M was not the supremum after all.

Contradiction. $\square$

Next

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f is **integrable**.

The goal is to find step functions

$$s(x) \leq f(x) \leq t(x)$$

with $\int_a^b s$ and $\int_a^b t$ as close together as we like — i.e. so that the gap between the two staircases is uniformly small.

That phrase “uniformly” is the thing we need to make precise.

Uniform continuity

Definition

$f: S \rightarrow \mathbb{R}$ is **uniformly continuous** if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \epsilon.$$

Compare with ordinary continuity: there, δ is allowed to depend on the point. Here one δ has to work **everywhere at once**.