

MATH 1610 / HONORS CALCULUS

Honors Calculus 1610

Lecture 9: Uniform Continuity, Integrability, and the Derivative

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Today

- ◆ Finish the story about integration: every continuous function on a closed interval is integrable.
- ◆ The tool that makes it work: uniform continuity.
- ◆ Then begin differential calculus.

Where we stopped

We wanted to show that a continuous f on $[a, b]$ is integrable, by trapping it between two step functions

$$s \leq f \leq t$$

whose integrals are as close together as we like.

To build those step functions we need to cut $[a, b]$ into pieces on which f does not vary by much — and we need **one** choice of piece size that works for the whole interval at once.

Uniform continuity

Definition

Let $S \subseteq \mathbb{R}$. A function $f: S \rightarrow \mathbb{R}$ is **uniformly continuous** if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \epsilon$$

for all $x_1, x_2 \in S$.

What actually moved

Continuity at every point of S :

$$\forall c \quad \forall \epsilon > 0 \quad \exists \delta > 0 : |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Uniform continuity on S :

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \forall x_1, x_2 \quad |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \epsilon.$$

In the first, the point is chosen **before** δ , so δ is allowed to depend on it. In the second the points are not chosen until **after** δ has been fixed — so one δ has to work **everywhere at once**.

The theorem

Theorem

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f is uniformly continuous.

The hypothesis that the interval is **closed and bounded** is essential — the same place it was essential for boundedness and for the extreme value theorem. The proof is the same bisection argument.

Good and bad intervals

Fix $\epsilon > 0$ for the whole proof.

Definition (for this proof)

An interval $I \subseteq [a, b]$ is **bad** if for every $\delta > 0$ there are $x_1, x_2 \in I$ with

$$|x_1 - x_2| < \delta \quad \text{and} \quad |f(x_1) - f(x_2)| \geq \epsilon,$$

and **good** otherwise — that is, if some single δ works throughout I .

Suppose the theorem fails. Then for some ϵ no δ works, which says exactly that $[a, b]$ itself is bad.

The key claim

Claim

If an interval I is bad, then one of its two halves is bad.

Equivalently: if both halves are good, so is I . That is what we prove. Write r for the midpoint of I .

There are three ways two points of I can sit relative to r , and we handle them separately.

Proof of the claim

Suppose both halves are good: there are δ_L and δ_R that work on the left and right halves.

Since f is **continuous at r** , there is $\delta_1 > 0$ with

$$|x - r| < \delta_1 \implies |f(x) - f(r)| < \frac{\epsilon}{2}.$$

Let $\delta = \min \{\delta_L, \delta_R, \delta_1\}$, and take $x_1, x_2 \in I$ with $|x_1 - x_2| < \delta$.

The three cases

- ◆ Both x_1, x_2 in the **left** half: then $|f(x_1) - f(x_2)| < \epsilon$, since $\delta \leq \delta_L$.
- ◆ Both in the **right** half: likewise, since $\delta \leq \delta_R$.
- ◆ They **straddle** r . Then r lies between them, so

$$|x_1 - r| \leq |x_1 - x_2| < \delta \leq \delta_1,$$

and the same for x_2 . Hence

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(r)| + |f(r) - f(x_2)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So this one δ works throughout I : I is good. $\square$

Rinse and repeat

Start with $I_0 = [a, b]$, which is bad. By the claim one half is bad; call it I_1 . Repeat:

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots, \quad \text{each } I_n \text{ bad, of length } \frac{b-a}{2^n}.$$

Let x_n be the left endpoint of I_n . These increase and are bounded above by b , so we may set

$$c = \sup \{x_0, x_1, x_2, \dots\}.$$

Since the intervals are nested and closed, $c \in I_n$ for every n .

A good interval around c

Apply continuity at c with $\epsilon/2$: there is $h > 0$ with

$$|x - c| < h \implies |f(x) - f(c)| < \frac{\epsilon}{2}.$$

So for **any** two points $x_1, x_2 \in (c - h, c + h)$,

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(c)| + |f(c) - f(x_2)| < \epsilon.$$

In particular $(c - h, c + h)$ is good — and so is every subinterval of it.

The contradiction

Choose n large enough that

$$\frac{b-a}{2^n} < \frac{h}{2}.$$

Then I_n contains c and has length less than $h/2$, so

$$I_n \subseteq (c-h, c+h).$$

But I_n is bad and $(c-h, c+h)$ is good, and a subinterval of a good interval is good.

Contradiction.

So no such ϵ exists, and f is not uniformly continuous. $\square$

The payoff

Corollary

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then f is integrable.

Recall the two numbers

$$U = \inf \left\{ \int_a^b t \mid t \text{ a step function, } t \geq f \right\},$$

$$L = \sup \left\{ \int_a^b s \mid s \text{ a step function, } s \leq f \right\},$$

and that $L \leq U$ always. Integrable means $L = U$.

A small principle, used twice

We will show $|L - U| < \epsilon$ for every $\epsilon > 0$. That forces $|L - U| = 0$.

Why: suppose $|L - U| \neq 0$. Then $|L - U|/2$ is itself a positive number, so we may take $\epsilon = |L - U|/2$ and conclude

$$|L - U| < \frac{|L - U|}{2},$$

which is false. So $|L - U| = 0$, i.e. $L = U$.

Building the two step functions

Let $\epsilon > 0$. Since f is **uniformly** continuous, choose $\delta > 0$ with

$$|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \frac{\epsilon}{b-a}.$$

Subdivide into n equal pieces,

$$a = x_0 < x_1 < \cdots < x_n = b, \quad x_i - x_{i-1} = \frac{b-a}{n},$$

choosing n large enough so that $\frac{b-a}{n} < \delta$.

The heights, and why they are close

On the i -th piece set

$$m_i = \min \{f(x) \mid x \in [x_{i-1}, x_i]\} = f(d_i),$$

$$M_i = \max \{f(x) \mid x \in [x_{i-1}, x_i]\} = f(c_i).$$

Both are **attained**, by the extreme value theorem.

Now c_i and d_i lie in the same piece, so $|c_i - d_i| < \delta$, and therefore

$$M_i - m_i = |f(c_i) - f(d_i)| < \frac{\epsilon}{b - a}.$$

Let s be the step function with value m_i on the i -th piece and t the one with value M_i . Then $s \leq f \leq t$.

The estimate

$$\int_a^b t - \int_a^b s = \int_a^b (t - s) = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^n (M_i - m_i) \frac{b-a}{n},$$

and since each $M_i - m_i < \frac{\epsilon}{b-a}$,

$$< \sum_{i=1}^n \frac{\epsilon}{b-a} \cdot \frac{b-a}{n} = \frac{\epsilon}{n} \sum_{i=1}^n 1 = \epsilon.$$

Conclusion

Since $s \leq f \leq t$ are step functions,

$$\int_a^b s \leq L \leq U \leq \int_a^b t,$$

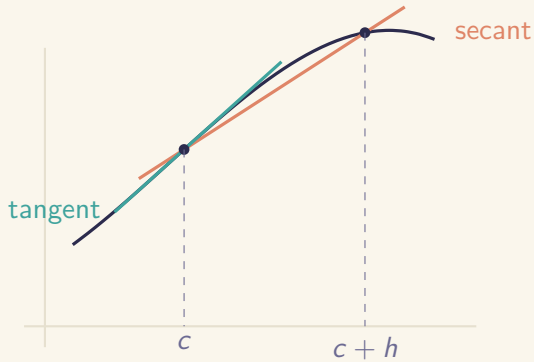
so

$$|L - U| \leq \int_a^b t - \int_a^b s < \epsilon.$$

This holds for every $\epsilon > 0$, so by the small principle $L = U$: the function f is integrable. $\square$

Differential calculus

Secants and tangents



Secant and tangent

Definition

A **secant line** to a graph is a line joining two points on the graph.

The **tangent line** at a point is the limit of the secant lines through that point, as the two points come together.

The secant through $(c, f(c))$ and $(c + h, f(c + h))$ has slope

$$\frac{f(c + h) - f(c)}{(c + h) - c} = \frac{f(c + h) - f(c)}{h},$$

the **difference quotient**. The slope of the tangent is what we will call the derivative.

The derivative

Definition

The **derivative** of f at $x = c$ is

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h},$$

if the limit exists. When it does, we say f is **differentiable** at c .

Note that the difference quotient is not even **defined** at $h = 0$ — it reads $0/0$ there. This is exactly why our definition of limit carries the $0 < |h|$.

Unwinding the limit

Written out, $f'(c) = \lambda$ means: for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < |h| < \delta \implies \left| \frac{f(c+h) - f(c)}{h} - \lambda \right| < \epsilon.$$

The limit is taken as $h \rightarrow 0$, and the $0 < |h|$ means we never ask what happens **at** $h = 0$ — which is fortunate, because nothing happens there.

Example: $f(x) = 3x + 1$

$$\frac{f(c+h) - f(c)}{h} = \frac{(3(c+h) + 1) - (3c + 1)}{h} = \frac{3c + 3h + 1 - 3c - 1}{h} = \frac{3h}{h}.$$

So the whole question is whether

$$\lim_{h \rightarrow 0} \frac{3h}{h}$$

exists, and what it is.

The check

Claim

$$\lim_{h \rightarrow 0} \frac{3h}{h} = 3.$$

Given $\epsilon > 0$, we must produce $\delta > 0$ with

$$0 < |h - 0| < \delta \implies \left| \frac{3h}{h} - 3 \right| < \epsilon.$$

But when $0 < |h|$ we have $h \neq 0$, so $\frac{3h}{h} = 3$ honestly, and the quantity on the right is $|3 - 3| = 0 < \epsilon$.

Any δ at all will do. Hence $f'(c) = 3$ for every c .

Next time

- ◆ $f(x) = x^2$, and then x^n by the binomial theorem.
- ◆ Differentiable $\implies$ continuous (and why the converse fails, at $|x|$).
- ◆ The rules: sums, scalar multiples, Leibniz, reciprocals, quotients.
- ◆ The chain rule.