

Math 1610 · Honors Calculus I · Fall 2026

# The Zoo of Derivatives

*grad, curl, div, and their fundamental theorems*

Danny Krashen · torsorlab

# Contents


The zoo . . . . . 4

1 Divergence . . . . . 6

2 The zoo in three dimensions . . . . . 9

3 Twice is nothing . . . . . 12

4 Integrals over curves, regions and surfaces . . . . . 14

5 One fundamental theorem, five costumes . . . . . 17

6 Epilogue: it is all one derivative . . . . . 22

*It's déjà vu all over again.*

— Yogi Berra

*A rose is a rose is a rose is a rose.*

— Gertrude Stein

# The zoo

*Grad, then curl, then div:*
*each one sends the last to naught.*
*Edges hold the sum.*

In $\mathbb{R}^3$ there are three kinds of derivative that turn one kind of object into another, and they line up in a row:

$$\text{functions} \xrightarrow{\nabla} \text{vector fields} \xrightarrow{\text{curl}} \text{vector fields} \xrightarrow{\text{div}} \text{functions}.$$

The gradient and the curl are old friends from the notes *Grad and Curl with Matrices and a Bit of Linear Algebra*. The new animal is the *divergence*. Write $\nabla$ for the list of partial derivative symbols $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. The divergence of a vector field $\vec{G} = (G_1, G_2, G_3)$ is the "dot product" of $\nabla$ with $\vec{G}$:

$$\operatorname{div} \vec{G} = \nabla \cdot \vec{G} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}.$$

It is a single function, and it measures how much $\vec{G}$ spreads out from each point. These notes make four claims about the row.

1. **Divergence measures sources.** If $\vec{G}$ is the velocity of a fluid, then $\operatorname{div} \vec{G}$ at a point is the rate at which fluid flows out of a tiny region around the point, per unit of volume.

2. **Two steps in a row give zero.** The curl of a gradient is zero, and the divergence of a curl is zero. The first says that gradients have zero curl. The second is new, and Chapter 3 proves it. Both are the equality of mixed partial derivatives in disguise.

3. **Each derivative has its own fundamental theorem of calculus.** Integrat-

ing a derivative over a region gives the same answer as integrating the original object over the boundary of the region. For an interval this is the familiar $\int_a^b f'(t)\,dt = f(b) - f(a)$. For a curve it is the theorem that line integrals of gradients depend only on the endpoints. For a region in the plane it is Green's theorem; for a surface in $\mathbb{R}^3$ it is Stokes's theorem; for a solid in $\mathbb{R}^3$ it is the divergence theorem.

4. **It is all one story.** In the last chapter, a new notation, *differential forms*, turns grad, curl and div into a single operation $d$, and turns the five theorems into the single formula $\int_M d\omega = \int_{\partial M} \omega$, Stokes's theorem in its general form. Nothing before the last chapter depends on it.

The first two claims are proved here. The integral theorems are stated but not proved. They are worth having anyway: five theorems that come with five different names and five different pictures turn out to be one theorem wearing five costumes, which is the sort of economy that mathematics is always hoping for and rarely gets.

Chapter 1

# Divergence

*Dot the nabla with G, and you'll find*
*Every slope on the diagonal lined;*
*Add them up: that's the trace,*
*And it tells, in each place,*
*How much fluid flows out, unconfined.*

The divergence of a vector field $\vec{G}$ on $\mathbb{R}^n$ is the sum $\partial G_1/\partial x_1 + \cdots + \partial G_n/\partial x_n$, which can be remembered as $\nabla \cdot \vec{G}$ and computed as the trace of the matrix $D\vec{G}$. It measures the outward flow of $\vec{G}$ from each point, per unit of volume.

## 1.1 The definition

The gradient of a function $f$ on $\mathbb{R}^n$ is the vector of its partial derivatives, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$. It is convenient to pretend that the symbol $\nabla$ is itself a vector,

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right),$$

whose entries are instructions rather than numbers, and that "multiplying" the instruction $\partial/\partial x_i$ by a function means carrying it out, that is, taking the partial derivative. Then $\nabla f$ is the vector $\nabla$ multiplied by the number $f$, entry by entry. The same pretense suggests what $\nabla$ should do to a vector field: take a dot product.

**Definition 1.1.** Let $\vec{G} = (G_1, \ldots, G_n)$ be a vector field on an open set in $\mathbb{R}^n$ whose entries have partial derivatives. The *divergence* of $\vec{G}$ is the function

$$\operatorname{div} \vec{G} = \nabla \cdot \vec{G} = \frac{\partial G_1}{\partial x_1} + \cdots + \frac{\partial G_n}{\partial x_n}.$$

Each entry of $\vec{G}$ is differentiated in its own direction: $G_1$ in the $x_1$-direction, $G_2$

in the $x_2$-direction, and so on, and the results are added. The mixed partials, like $\partial G_1/\partial x_2$, do not appear at all.

The pretense has limits. The symbol $\nabla$ is not a vector, and $\nabla \cdot \vec{G}$ is not the same as $\vec{G} \cdot \nabla$. The first is a function; the second, $G_1 \frac{\partial}{\partial x_1} + \cdots + G_n \frac{\partial}{\partial x_n}$, is an instruction still waiting for something to differentiate. Dot products of honest vectors do not behave like this. Treat $\nabla \cdot \vec{G}$ as a mnemonic for the formula, not as an actual dot product.

**Divergence is a trace.** The derivative matrix $D\vec{G}$ has $(i, j)$ entry $\partial G_i/\partial x_j$, so its diagonal entries are exactly the terms in the divergence. The sum of the diagonal entries of a square matrix is called its *trace*, so

$$\operatorname{div} \vec{G} = \operatorname{tr}(D\vec{G}).$$

Recall that $D\vec{G}$ splits into a symmetric part and an antisymmetric part, and that the curl only sees the antisymmetric part. The divergence only sees the symmetric part, because an antisymmetric matrix has zeros on its diagonal. The two derivatives look at opposite halves of the same matrix.

**Example 1.2.** (a) The radial field $\vec{G} = (x, y, z)$ has $\operatorname{div} \vec{G} = 1 + 1 + 1 = 3$. Every point pushes outward.

(b) The swirl $\vec{G} = (-y, x, 0)$ has $\operatorname{div} \vec{G} = 0 + 0 + 0 = 0$. Things go round and round and nothing is created; so it goes.

(c) For a linear field $\vec{G}(\vec{x}) = A\vec{x}$, $D\vec{G} = A$, so $\operatorname{div} \vec{G} = \operatorname{tr} A$. The swirl's matrix has zero trace; the radial field's matrix is the identity, with trace 3.

(d) $\vec{G} = (x^2 y,\ yz,\ xz^2)$ has $\operatorname{div} \vec{G} = 2xy + z + 2xz$.

## 1.2 What divergence measures

Think of $\vec{G}$ as the velocity of a fluid. The divergence at a point is the rate at which fluid flows *out* of a tiny region around that point, per unit of volume (or of area, in the plane). Positive divergence means a source, where fluid appears (a spring, a tap, a leak in the universe). Negative divergence means a sink. Zero divergence means that whatever flows in also flows out.

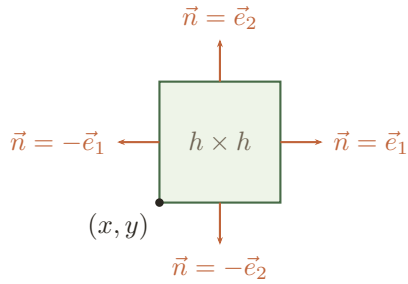
How much fluid crosses a short straight edge? Only the part of the velocity

perpendicular to the edge carries fluid across it; the part along the edge just slides by. So if $\vec{n}$ is a unit vector perpendicular to the edge, pointing in the direction we count as "out", the flow across the edge per unit time is about

$$(\vec{G} \cdot \vec{n}) \times (\text{length of the edge}).$$

This is a dot product of the vector-valued function $\vec{G}$ with a fixed vector, and it is the honest dot product that the notation $\nabla \cdot \vec{G}$ is imitating.

Here is the computation behind the claim, for a tiny square in the plane, in the same spirit as the computation that made the scalar curl the circulation per unit area. It is a heuristic: it uses approximations without controlling the error. The divergence theorem in Chapter 5 is the exact version.




Take the square with corners $(x, y)$ and $(x + h, y + h)$, and count outward flow as positive. On the right edge the outward normal is $\vec{e}_1$, so the flow out is about $G_1(x + h, y)\, h$. On the left edge it is $-\vec{e}_1$, so the flow out is about $-G_1(x, y)\, h$. The top and bottom edges contribute about $G_2(x, y + h)\, h$ and $-G_2(x, y)\, h$. The total is about

$$\begin{aligned} &h\big(G_1(x + h, y) - G_1(x, y)\big) + h\big(G_2(x, y + h) - G_2(x, y)\big) \\ &\qquad \approx h^2\Big(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}\Big), \end{aligned}$$

which is the divergence times the area $h^2$. The left and right edges measure how $G_1$ changes in the $x$-direction, and the top and bottom measure how $G_2$ changes in the $y$-direction. That is why each entry of $\vec{G}$ is differentiated only in its own direction.

Chapter 2

# The zoo in three dimensions

*In three-space the ladder is neat:*
*Function, field, field, function: complete.*
*Climb by grad, curl, and div,*
*As the binomials give,*
*Where the threes of Pascal's triangle meet.*

In $\mathbb{R}^3$, the three derivatives can all be written with $\nabla$:

$$\nabla f, \qquad \nabla \times \vec{F} = \operatorname{curl} \vec{F}, \qquad \nabla \cdot \vec{G} = \operatorname{div} \vec{G}.$$

The gradient multiplies $\nabla$ by a function, the curl takes a cross product with $\nabla$, and the divergence takes a dot product. This works only in $\mathbb{R}^3$, because only there is the curl a vector.

## 2.1 Three products with nabla

Recall the cross product of $\vec{v}, \vec{w} \in \mathbb{R}^3$,

$$\vec{v} \times \vec{w} = \big(v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1\big).$$

Put $\nabla = (\partial_x, \partial_y, \partial_z)$ in place of $\vec{v}$ and a field $\vec{F} = (P, Q, R)$ in place of $\vec{w}$, and let "multiplying" mean differentiating:

$$\nabla \times \vec{F} = \big(R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\big).$$

This is exactly the curl vector from the grad-and-curl notes. The same computation is often packaged as a determinant, expanded along the top row, with

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ standing in the first row:

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix} = \vec{e}_1(R_y - Q_z) - \vec{e}_2(R_x - P_z) + \vec{e}_3(Q_x - P_y).$$

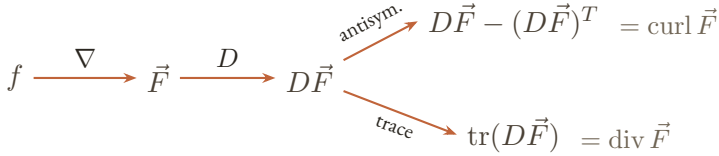
Like $\nabla$ itself, this "determinant" is a mnemonic: its entries are not all numbers. So the three derivatives of $\mathbb{R}^3$ are three ways of multiplying by $\nabla$:

| derivative | input | output | formula |
|---|---|---|---|
| gradient | function $f$ | vector field | $\nabla f = (f_x, f_y, f_z)$ |
| curl | vector field $\vec{F}$ | vector field | $\nabla \times \vec{F}$ |
| divergence | vector field $\vec{G}$ | function | $\nabla \cdot \vec{G}$ |

**Example 2.1.** For $\vec{F} = (yz,\ 0,\ x^2)$: $\nabla \times \vec{F} = (0 - 0,\ y - 2x,\ 0 - z) = (0,\ y - 2x,\ -z)$ and $\nabla \cdot \vec{F} = 0 + 0 + 0 = 0$.

## 2.2 All the derivatives from one matrix

In any dimension, the curl and the divergence are both read off from the derivative matrix $D\vec{F}$, which collects every first partial derivative of $\vec{F}$:




When $\vec{F} = \nabla g$, the derivative matrix $D(\nabla g)$ has entries $\partial^2 g / \partial x_j \partial x_i$; it is called the *Hessian* of $g$ and written $H_g$. So $H_g - H_g^T$ is the curl of $\nabla g$, and the trace of $H_g$ is the divergence of $\nabla g$.

## 2.3 Why only in three dimensions

In $\mathbb{R}^n$, the gradient of a function always has $n$ entries, and the divergence of a vector field is always a single function, so both make sense in every dimension. The curl is different. The curl of a field on $\mathbb{R}^n$ is the antisymmetric matrix $D\vec{F} - (D\vec{F})^T$, which is determined by its $\frac{n(n-1)}{2}$ entries above the diagonal. That number equals $n$ only when $n = 3$. In $\mathbb{R}^2$ the curl is one number, the

scalar curl; in $\mathbb{R}^4$ it is six numbers, which is neither a function nor a vector field. In $\mathbb{R}^3$, the curl is three numbers, a vector field, and so the row of derivatives can keep going: function, field, field, function. (Those counts $1, 3, 3, 1$ are a row of Pascal's triangle, which the last chapter will explain.) Physics in three dimensions is written with vectors because of this coincidence. It is the kind of luck the universe rarely extends twice.

**Exercise 2.2.** For $\vec{F} = (xy,\ yz,\ zx)$, compute $\nabla \times \vec{F}$ and $\nabla \cdot \vec{F}$.

Chapter 3

# Twice is nothing

*Take the curl of a grad, and you've got*
*Not a thing; div of curl: the same lot.*
*It's no twist of fate:*
*Mixed partials equate,*
*And they cancel in pairs on the spot.*

**Theorem 3.1.** *On an open set in* $\mathbb{R}^3$*:*

*(a) if* $f$ *has continuous second partial derivatives, then* $\operatorname{curl}(\nabla f) = \vec{0}$*;*

*(b) if* $\vec{F}$ *has continuous second partial derivatives, then* $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$*.*

*Proof.* Part (a) is the theorem that gradients have zero curl, from the grad-and-curl notes: the derivative matrix of $\nabla f$ is the Hessian $H_f$, which is symmetric, so the curl matrix $H_f - H_f^T$ is zero. Entry by entry, it has entries $f_{x_j x_i} - f_{x_i x_j}$, which are zero because mixed partial derivatives agree. For part (b), let $\vec{F} = (P, Q, R)$. Then

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y).$$

Expanding gives six terms:

$$R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}.$$

(Here $R_{yx}$ means: first differentiate $R$ in $y$, then in $x$.) The terms cancel in pairs, $R_{yx}$ with $R_{xy}$, $Q_{zx}$ with $Q_{xz}$, and $P_{zy}$ with $P_{yz}$, because mixed partial derivatives of functions with continuous second partials agree. □

In $\nabla$ notation the theorem says $\nabla \times (\nabla f) = \vec{0}$ and $\nabla \cdot (\nabla \times \vec{F}) = 0$. For honest vectors, the corresponding facts are $\vec{v} \times \vec{v} = \vec{0}$ and $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$,

both from the grad-and-curl notes. So the mnemonic predicts the theorem. It is not a proof, since $\nabla$ is not a vector; the proof is the cancellation of mixed partials above.

Only *consecutive* steps in the row cancel. The divergence of a gradient is not zero in general:

$$\nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz},$$

which is 2 for $f = x^2$. (This combination is called the *Laplacian* of $f$, and it is a celebrity in its own right, but it belongs to a different zoo.)

**A test, and a converse.** Theorem 3.1(b) is a test: if $\operatorname{div} \vec{G} \neq 0$ anywhere, then $\vec{G}$ is not the curl of any field with continuous second partials. The radial field $(x, y, z)$, with divergence 3, is not a curl. As with gradients, there is a converse on a ball, which these notes state without proof: if $\vec{G}$ has continuous partial derivatives on an open ball in $\mathbb{R}^3$ and $\operatorname{div} \vec{G} = 0$ there, then $\vec{G} = \operatorname{curl} \vec{F}$ for some field $\vec{F}$ on the ball. And as with gradients, the converse fails on regions with holes, this time holes you can surround with a sphere (Exercise 5.15).

**Exercise 3.2.** For the field $\vec{F}$ of Exercise 2.2, check directly that $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$. Is $\vec{G} = (x,\ -2y,\ z)$ allowed to be a curl by the test? Is $\vec{G} = (x,\ y,\ z)$?

Chapter 4

# Integrals over curves, regions and surfaces

*On a curve, we dot $F$ with the way;*
*On a surface, $G \cdot n$ holds sway;*
*Through each patch of the sheet*
*Count the flow, and repeat;*
*Turn it round, and a minus will stay.*

Each derivative in the zoo comes with its own kind of integral. A gradient or a curl, thought of as a field to be pushed *along* something, is integrated along a curve. A curl or any other field thought of as a flow *through* something is integrated over a surface, as a flux. A divergence, being a function, is integrated over a region of the plane or a solid in space. Each kind of region comes with an *orientation*, a choice of direction, and reversing the orientation changes the sign of the integral.

## 4.1 Along curves

An oriented curve $C$ is a curve with a direction of travel. It is described by a parametrization $\vec{r}(t)$, $a \le t \le b$, that moves in that direction. The line integral of a field $\vec{F}$ along $C$ is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$$

Running the curve backwards changes its sign; that is the rule about orientation in its first appearance. When $C$ is a closed curve, the line integral is often written $\oint_C$ and called the *circulation* of $\vec{F}$ around $C$.

**Flux across a curve in the plane.** In the plane there is a second integral along a curve, which measures flow *across* it instead of along it. Suppose $C$ is a closed curve in $\mathbb{R}^2$, traversed by $\vec{r}(t) = (x(t), y(t))$ so that the region it encloses is on the left. Turn the velocity $\vec{r}'(t) = (x', y')$ a quarter turn clockwise: the result,

$(y', -x')$, has the same length as the velocity and points out of the region. The *outward flux* of $\vec{G}$ across $C$ is

$$\oint_C \vec{G} \cdot \vec{n}\, ds = \int_a^b \vec{G}(\vec{r}(t)) \cdot \big(y'(t), -x'(t)\big)\, dt.$$

The notation records the meaning: $\vec{n}$ is the outward unit normal and $ds$ is length along the curve, so the integrand is the outward component of the flow times length, exactly as for the edges of the tiny square.

## 4.2 Over regions and surfaces

**Regions in the plane, and solids.** For a function $g$ on a region $D$ in $\mathbb{R}^2$, the integral is the double integral $\iint_D g\, dA$. For a function on a solid region $E$ in $\mathbb{R}^3$, it is the triple integral $\iiint_E g\, dV$, defined with boxes in place of rectangles exactly as the double integral is.

**Surfaces in space.** A parametrized surface is a function $\vec{r}(u, v)$, with values in $\mathbb{R}^3$, defined for $(u, v)$ in a region $D$ of the plane. A tiny $h$-by-$h$ rectangle in $D$, at the point $(u, v)$, is carried by $\vec{r}$ to a tiny patch on the surface that is approximately the parallelogram spanned by $h\,\vec{r}_u$ and $h\,\vec{r}_v$. Two facts about the cross product, the first from the grad-and-curl notes and the second stated without proof here:

- $\vec{r}_u \times \vec{r}_v$ is perpendicular to both $\vec{r}_u$ and $\vec{r}_v$, so it is perpendicular to the surface;
- the length of $\vec{v} \times \vec{w}$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

So $h^2(\vec{r}_u \times \vec{r}_v)$ is a vector perpendicular to the patch whose length is the patch's area. The flow of $\vec{G}$ through the patch is $(\vec{G} \cdot \vec{n})$ times the area, where $\vec{n}$ is the unit normal, and that is $\vec{G} \cdot (\vec{r}_u \times \vec{r}_v)\, h^2$. Adding up over all the patches suggests the following definition.

**Definition 4.1.** Let $\vec{r}$ be a parametrized surface $S$ with parameter region $D$, and let $\vec{G}$ be a continuous vector field on $S$. The *flux* of $\vec{G}$ through $S$ is

$$\iint_S \vec{G} \cdot \vec{n}\, dA = \iint_D \vec{G}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$

The surface is oriented by the normal vector $\vec{r}_u \times \vec{r}_v$.

The flux is how much fluid crosses $S$ per unit time, counted positively in the direction of $\vec{r}_u \times \vec{r}_v$. Swapping $u$ and $v$ reverses the normal and the sign; reversing orientation flips the sign, as always. Changing to a different parametrization of the same surface with the same orientation does not change the flux; these notes do not prove that. A closed surface, such as the boundary of a solid, is oriented by the normal that points out of the solid.

**Example 4.2.** The unit sphere is parametrized by

$$\vec{r}(\varphi, \theta) = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi),$$

for $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$, where $\varphi$ is the angle down from the north pole and $\theta$ the angle around the $z$-axis. A computation gives

$$\vec{r}_\varphi \times \vec{r}_\theta = \sin\varphi\ \vec{r}(\varphi, \theta),$$

which points outward, since $\sin\varphi \ge 0$. For the radial field $\vec{G} = (x, y, z)$, at the point $\vec{r}$ we have $\vec{G} = \vec{r}$, and $\vec{r} \cdot \vec{r} = 1$, so the integrand is $\sin\varphi$, and the flux out of the sphere is

$$\int_0^{2\pi}\int_0^{\pi} \sin\varphi\, d\varphi\, d\theta = 2\pi \cdot \big[-\cos\varphi\big]_0^{\pi} = 4\pi.$$

This is the surface area of the sphere, as it should be: $\vec{G}$ is the unit outward normal at every point of the sphere, so it pushes one unit of fluid through each unit of area.

**Exercise 4.3.** (a) Compute the circulation of $\vec{F} = (y, -x)$ around the unit circle, traversed counterclockwise. Then compute the outward flux of the same field across the unit circle.

(b) Check the formula for $\vec{r}_\varphi \times \vec{r}_\theta$ in Example 4.2.

(c) Find the flux of $\vec{G} = (0, 0, 1)$ upward through the upper half of the unit sphere, using the same parametrization with $0 \le \varphi \le \pi/2$. Why is the answer the area of a disk of radius 1? (A function whose derivative is $\sin\varphi\cos\varphi$ is $\frac{1}{2}\sin^2\varphi$.)

Chapter 5

# One fundamental theorem, five costumes

*Integrate a derivative in space,*
*And the inside drops out without trace:*
*Only edges remain.*
*It's the same theorem again,*
*Wearing Green's, Stokes's, or Gauss's face.*

Each of the five theorems in this chapter says the same thing: the integral of a derivative over a region equals an integral over the boundary of the region of the thing that was differentiated. The interior of the region contributes nothing beyond what the edge already knows. These notes state the theorems and do not prove them.

| region | its boundary | derivative | name |
|---|---|---|---|
| interval | 2 endpoints | $f'$ | FTC |
| curve | 2 endpoints | $\nabla f$ | gradient theorem |
| plane region | closed curves | scalar curl or div | Green |
| surface in $\mathbb{R}^3$ | closed curve | curl | Stokes |
| solid in $\mathbb{R}^3$ | closed surface | div | divergence |

The one-variable theorem you learned first was this whole story already; it just had very little room to show off.

## 5.1 Dimension one

**Theorem 5.1** (fundamental theorem of calculus). *If $f$ has a continuous derivative on $[a, b]$, then $\int_a^b f'(t)\,dt = f(b) - f(a)$.*

**Theorem 5.2** (gradient theorem). *If $f$ has continuous partial derivatives on an open set $U$ in $\mathbb{R}^n$, and $C$ is a piecewise smooth curve in $U$ from $\vec{p}$ to $\vec{q}$, then $\int_C \nabla f \cdot d\vec{r} = f(\vec{q}) - f(\vec{p})$.*

In both, the boundary of the region is two points, the end counted with a plus sign and the start with a minus sign.

## 5.2 Dimension two: Green's theorem, twice

**Theorem 5.3** (Green's theorem). *Let $D$ be a bounded region in $\mathbb{R}^2$ whose boundary $\partial D$ consists of finitely many piecewise smooth simple closed curves, oriented so that $D$ is on the left. Let $\vec{F} = (P, Q)$ have continuous partial derivatives on an open set containing $D$ and $\partial D$. Then*

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA.$$

In words: the circulation of $\vec{F}$ around the edge of $D$ is the total scalar curl inside. That fits the picture of the scalar curl as circulation per unit area: chop $D$ into tiny squares, and the circulations around neighboring squares cancel along every shared edge, because the edge is traversed once in each direction. Only the outer edge survives.

**Example 5.4.** For the swirl $\vec{F} = (-y, x)$ on the unit disk, the scalar curl is $Q_x - P_y = 2$, so the right side is 2 times the area, $2\pi$. On the left, with $\vec{r}(t) = (\cos t, \sin t)$,

$$\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

Turning every arrow a quarter turn converts circulation into flux and scalar curl into divergence, and Green's theorem comes along for the ride.

**Corollary 5.5** (Green's theorem, flux form). *Under the hypotheses of Theorem 5.3, for a vector field $\vec{G}$ with continuous partial derivatives on an open set containing $D$ and $\partial D$,*

$$\oint_{\partial D} \vec{G} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{G}\, dA.$$

*Proof.* Let $\vec{F} = (-G_2, G_1)$, which is $\vec{G}$ turned a quarter turn counterclockwise. Along the boundary,

$$\vec{F} \cdot \vec{r}\,' = -G_2 x' + G_1 y' = \vec{G} \cdot (y', -x'),$$

so the circulation of $\vec{F}$ is the outward flux of $\vec{G}$. And the scalar curl of $\vec{F}$ is $\partial_x G_1 - \partial_y(-G_2) = \operatorname{div} \vec{G}$. Now apply Theorem 5.3 to $\vec{F}$. □

The flow out across the edge of $D$ is the total of the sources inside. This is the tiny-square heuristic of Chapter 1, now with an equals sign.

Remark 5.6 (the hole, again). Green's theorem explains the angle field $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ from the grad-and-curl notes. Its scalar curl is 0 wherever it is defined, yet its integral around the unit circle is $2\pi$. Green's theorem does not apply to the unit disk, because $\vec{F}$ is not defined at the origin, which is inside the disk. If a closed curve $C$ is the boundary of a region on which $\vec{F}$ is defined, Green's theorem gives $\oint_C \vec{F} \cdot d\vec{r} = \iint 0 \, dA = 0$. The unit circle is not such a boundary: the region it surrounds contains the hole.

## 5.3 Dimension three: Stokes and Gauss

**Theorem 5.7** (Stokes's theorem). *Let $D$ be a region as in Theorem 5.3, and let $\vec{r}(u, v)$ have continuous second partial derivatives on an open set containing $D$ and $\partial D$. Let $S$ be the parametrized surface $\vec{r}$ on $D$, and let $\partial S$ be the curve traced by $\vec{r}$ as $(u, v)$ runs around $\partial D$ with $D$ on the left. If $\vec{F}$ has continuous partial derivatives on an open set containing $S$, then*

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n} \, dA.$$

In words: the circulation of $\vec{F}$ around the edge of a surface is the flux of its curl through the surface. The orientation is the right-hand rule: if the thumb of your right hand points along $\vec{r}_u \times \vec{r}_v$, your fingers curl in the direction of $\partial S$. When the surface is a flat region in the $xy$-plane, with normal $\vec{e}_3$, the flux of the curl is the integral of its third entry $Q_x - P_y$, and Stokes's theorem becomes Green's.

**Example 5.8.** Take $\vec{F} = (-y, x, 0)$, whose curl is $(0, 0, 2)$, and the upper half of the unit sphere: the parametrization of Example 4.2 with $0 \le \varphi \le \pi/2$. As $(\varphi, \theta)$ goes around the rectangle $[0, \pi/2] \times [0, 2\pi]$, the image runs along the equator counterclockwise (seen from above), runs along one meridian down and back up (which cancels), and sits still at the north pole. So $\partial S$ is the equator,

and $\oint \vec{F} \cdot d\vec{r} = 2\pi$ as in the plane. On the other side,

$$(\operatorname{curl} \vec{F}) \cdot (\vec{r}_\varphi \times \vec{r}_\theta) = (0,0,2) \cdot \sin\varphi\, \vec{r} = 2\sin\varphi\cos\varphi,$$

and $\int_0^{2\pi}\int_0^{\pi/2} 2\sin\varphi\cos\varphi\, d\varphi\, d\theta = 2\pi\left[\sin^2\varphi\right]_0^{\pi/2} = 2\pi$. The flat unit disk has the same boundary, and the flux of $(0,0,2)$ up through it is $2 \cdot \pi = 2\pi$ too. Stokes's theorem says this had to happen: the flux of a curl through a surface depends only on the surface's edge.

**Theorem 5.9** (divergence theorem). *Let $E$ be a bounded solid in $\mathbb{R}^3$ whose boundary $\partial E$ is a piecewise smooth closed surface, oriented by the outward normal; a ball, a box and a solid cylinder all qualify. If $\vec{G}$ has continuous partial derivatives on an open set containing $E$ and $\partial E$, then*

$$\iint_{\partial E} \vec{G} \cdot \vec{n}\, dA = \iiint_E \operatorname{div} \vec{G}\, dV.$$

In words: the flow out through the skin of a solid is the total of the sources inside. It is also called Gauss's theorem.

**Example 5.10.** For $\vec{G} = (x,y,z)$ and the unit ball $E$, $\operatorname{div}\vec{G} = 3$, so the right side is 3 times the volume, $3 \cdot \frac{4}{3}\pi = 4\pi$. The left side is the flux out of the unit sphere, which was $4\pi$ in Example 4.2.

Remark 5.11 (the boundary of a boundary). The boundary of a solid ball is a sphere, and a sphere has no edge: the boundary of a boundary is empty. This geometric fact and the algebraic fact $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ are tied together by the theorems. For the unit ball $E$ and a field $\vec{F}$ with continuous second partials, the divergence theorem gives

$$\iint_{\partial E} \operatorname{curl}\vec{F} \cdot \vec{n}\, dA = \iiint_E \operatorname{div}(\operatorname{curl}\vec{F})\, dV = 0.$$

Stokes's theorem predicts the same zero by a different route. Split the sphere into its upper and lower halves, each oriented outward. By the right-hand rule, their edges are the same equator, traversed in opposite directions, so the two circulations cancel. The algebra and the geometry agree, which is reassuring, since otherwise one of them would have some explaining to do.

**Exercise 5.12.** Use Green's theorem with $\vec{F} = \frac{1}{2}(-y,\, x)$ to show that the area of a region $D$ is $\oint_{\partial D} \vec{F} \cdot d\vec{r}$. Use this to find the area inside the ellipse $(a\cos t,\ b\sin t)$, $0 \le t \le 2\pi$, with $a, b > 0$.

**Exercise 5.13.** Check the divergence theorem for $\vec{G} = (x, y, z)$ on the cube $[0,1] \times [0,1] \times [0,1]$, by computing the flux through each of the six faces. (On the face $x = 1$, the outward normal is $\vec{e}_1$.)

**Exercise 5.14.** On $\mathbb{R}^2 \setminus \{\vec{0}\}$, show that the field $\vec{G} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ has divergence $0$, and that its outward flux across the unit circle is $2\pi$. Why does this not contradict Corollary 5.5?

**Exercise 5.15.** On $\mathbb{R}^3 \setminus \{\vec{0}\}$, the field $\vec{G} = \vec{x}/\|\vec{x}\|^3$ has divergence $0$ (you may take this as given, or check it). Show that its flux out of the unit sphere is $4\pi$, and use the argument of Remark 5.11 to explain why $\vec{G}$ is not the curl of any field with continuous second partials on $\mathbb{R}^3 \setminus \{\vec{0}\}$.

Chapter 6

# Epilogue: it is all one derivative

一を以て万を知る事、兵法の利也。

*From one thing, know ten thousand things.*

— Miyamoto Musashi, *The Book of Five Rings*, the Ground Book

勿論空はなき也。有所を知りて無所を知る、是則空也。

*Of course the void is nothingness. By knowing things that exist, you can know that which does not exist.*

— Miyamoto Musashi, *The Book of Five Rings*, the Book of the Void

*You have to be somebody before you can be nobody.*

— Jack Engler

(Musashi translations by Victor Harris)

In the language of *differential forms*, the gradient, the curl and the divergence are a single operation $d$, called the *exterior derivative*; the two theorems of Chapter 3 become the single rule $d(d\omega) = 0$; and the five theorems of Chapter 5 become the single formula

$$\int_M d\omega = \int_{\partial M} \omega,$$

called *Stokes's theorem* in its general form. A $k$-form is the thing that sits under the integral sign when the region of integration is $k$-dimensional. The forms come in degrees $0, 1, \ldots, n$ on $\mathbb{R}^n$, and $d$ raises the degree by one.

## 6.1 One-forms

For a field $\vec{F} = (P, Q, R)$ and a curve $\vec{r}(t) = (x(t), y(t), z(t))$, the line integral is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \big(P\,x'(t) + Q\,y'(t) + R\,z'(t)\big)\,dt.$$

The same integral is also written

$$\int_C P\,dx + Q\,dy + R\,dz,$$

which is what you get from the formula above by writing $dx$ for $x'(t)\,dt$, $dy$ for $y'(t)\,dt$ and $dz$ for $z'(t)\,dt$, the same substitution rule as in one-variable integrals. The symbol $dx$ records how fast the curve moves in the $x$-direction, and similarly for $dy$ and $dz$. The expression $P\,dx + Q\,dy + R\,dz$ is called a 1-*form*. It carries the same information as the vector field $\vec{F}$, but it is written so as to be integrated along curves.

Here is another way to see what a 1-form is. At each point $\vec{x}$, the gradient of a function $f$ gives a *covector*, the linear function $\vec{h} \mapsto \nabla f(\vec{x}) \cdot \vec{h}$ that predicts how much $f$ changes when $\vec{x}$ moves by $\vec{h}$. A 1-form is a covector at each point: a covector-valued function. The 1-form that records the gradient of $f$ is written

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n.$$

More generally, any expression $f_1\,dg_1 + \cdots + f_r\,dg_r$, with smooth functions $f_i$ and $g_i$, is a 1-form: expand each $dg_i$ by the formula for $df$ and collect the coefficient of each $dx_j$. The product rule takes the form $d(fg) = f\,dg + g\,df$. In $\mathbb{R}^1$, for example, $dg = g'(x)\,dx$, so

$$f_1\,dg_1 + f_2\,dg_2 = \big(f_1\,g_1'(x) + f_2\,g_2'(x)\big)\,dx,$$

and every 1-form on $\mathbb{R}^1$ is a function times $dx$.

## 6.2 The rules

Higher forms are built by multiplying the symbols with a product written $\wedge$ and called the *wedge*. The wedge makes sense for vectors first. The *exterior algebra* (or *Grassmann algebra*) consists of formal sums of wedges of vectors, such as $\vec{v}_1 \wedge \vec{v}_2 \wedge \vec{v}_3 + \vec{v}_4 \wedge \vec{v}_5$, subject to the rules

$$\vec{v} \wedge \vec{w} = -\,\vec{w} \wedge \vec{v}, \qquad \lambda(\vec{v} \wedge \vec{w}) = (\lambda\vec{v}) \wedge \vec{w},$$
$$(\vec{v}_1 + \vec{v}_2) \wedge \vec{w} = \vec{v}_1 \wedge \vec{w} + \vec{v}_2 \wedge \vec{w}.$$

Taking $\vec{w} = \vec{v}$ in the first rule gives $\vec{v} \wedge \vec{v} = -\,\vec{v} \wedge \vec{v}$, so $\vec{v} \wedge \vec{v} = 0$.

**Example 6.1.** In $\mathbb{R}^2$, write $(a, b) = a\vec{e}_1 + b\vec{e}_2$ and $(c, d) = c\vec{e}_1 + d\vec{e}_2$. Distributing,

$$(a, b) \wedge (c, d) = ac\,\vec{e}_1 \wedge \vec{e}_1 + ad\,\vec{e}_1 \wedge \vec{e}_2 + bc\,\vec{e}_2 \wedge \vec{e}_1 + bd\,\vec{e}_2 \wedge \vec{e}_2.$$

The first and last terms are 0, and $\vec{e}_2 \wedge \vec{e}_1 = -\vec{e}_1 \wedge \vec{e}_2$, so

$$(a, b) \wedge (c, d) = (ad - bc)\,\vec{e}_1 \wedge \vec{e}_2.$$

The symbols $dx_1, \ldots, dx_n$ are wedged by the same rules, with functions allowed as coefficients.

**Definition 6.2** (the wedge rule for forms). For all $i$ and $j$,

$$dx_i \wedge dx_j = -\,dx_j \wedge dx_i.$$

Taking $i = j$ gives $dx_i \wedge dx_i = -\,dx_i \wedge dx_i$, so $dx_i \wedge dx_i = 0$.

So a product with a repeated symbol is zero, and a product of distinct symbols can be rearranged into increasing order at the cost of one sign for each swap. For example,

$$dz \wedge dx \wedge dy = -\,dx \wedge dz \wedge dy = dx \wedge dy \wedge dz,$$

two swaps, two signs, no net change. Signs that change whenever two things trade places should look familiar: they are the entries of an antisymmetric matrix, $k_{ij} = -k_{ji}$, and that resemblance is not an accident.

**Definition 6.3.** A *k-form* on an open set $U \subseteq \mathbb{R}^n$ is a sum

$$\omega = \sum g_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

one term for each choice of indices $i_1 < i_2 < \cdots < i_k$, where each coefficient $g_{i_1 \cdots i_k}$ is a function on $U$. A 0-form is just a function.

A $k$-form on $\mathbb{R}^n$ therefore has one coefficient for each way of choosing $k$ of the $n$ coordinates, which is $\binom{n}{k}$ coefficients. In $\mathbb{R}^3$:

| $k$ | a general $k$-form | coefficients |
|---|---|---|
| 0 | $f$ | 1 |
| 1 | $P\,dx + Q\,dy + R\,dz$ | 3 |
| 2 | $G_1\,dy \wedge dz + G_2\,dz \wedge dx + G_3\,dx \wedge dy$ | 3 |
| 3 | $g\,dx \wedge dy \wedge dz$ | 1 |

These are the counts $1, 3, 3, 1$ from Chapter 2. (For 2-forms the basic forms are listed in the cyclic order $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$; since $dz \wedge dx = -dx \wedge dz$, this only moves a sign into the coefficient, and it is what makes the curl come out right below.) There are no 4-forms on $\mathbb{R}^3$, because any product of four of $dx, dy, dz$ repeats one of them and is zero. The top-degree forms, the $n$-forms on $\mathbb{R}^n$, are also called *volume forms*.

**Example 6.4.** Multiplying two 1-forms on $\mathbb{R}^2$ is the same computation as Example 6.1, with $dx, dy$ in place of $\vec{e}_1, \vec{e}_2$:

$$(a\,dx + b\,dy) \wedge (c\,dx + e\,dy) = (ae - bc)\,dx \wedge dy.$$

The coefficient is the determinant of the matrix with rows $(a, b)$ and $(c, e)$, which is the signed area of the parallelogram spanned by those two vectors. A 2-form, it turns out, measures area, with a sign that remembers orientation.

**Exercise 6.5.** Simplify $(dx + 2\,dy) \wedge (3\,dx - dy)$ and $dy \wedge dx \wedge dy$. How many basic $k$-forms are there on $\mathbb{R}^4$, for $k = 0, 1, 2, 3, 4$?

## 6.3 The operation $d$

**Definition 6.6.** Let $U \subseteq \mathbb{R}^n$ be open.

(a) For a differentiable function $f$ on $U$,

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n.$$

(b) For a $k$-form $\omega = \sum g\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ with differentiable coefficients,

$$d\omega = \sum dg \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

That is, apply rule (a) to each coefficient and wedge the result on the front.

Rule (a) says that $df$ is the 1-form whose coefficients are the entries of $\nabla f$, so on functions $d$ is the gradient. Rule (b) is the only other rule, and the wedge rule does all the bookkeeping. Here is what it does in $\mathbb{R}^3$.

**Theorem 6.7.** *On an open set in $\mathbb{R}^3$, for differentiable $f$, $\vec{F} = (P, Q, R)$, $\vec{G} = (G_1, G_2, G_3)$ and $g$:*

$$\begin{aligned}
&df = f_x\,dx + f_y\,dy + f_z\,dz,\\
&d(P\,dx + Q\,dy + R\,dz)\\
&\qquad = (R_y - Q_z)\,dy \wedge dz + (P_z - R_x)\,dz \wedge dx + (Q_x - P_y)\,dx \wedge dy,\\
&d(G_1\,dy \wedge dz + G_2\,dz \wedge dx + G_3\,dx \wedge dy) = (\operatorname{div} \vec{G})\,dx \wedge dy \wedge dz,\\
&d(g\,dx \wedge dy \wedge dz) = 0.
\end{aligned}$$

*So, matching each form with the function or vector field of its coefficients, $d$ is the gradient, then the curl, then the divergence.*

*Proof.* The first line is rule (a). The last holds because the result would be a 4-form, and there are none. For the second, rule (b) gives

$$dP \wedge dx + dQ \wedge dy + dR \wedge dz,$$

and $dP \wedge dx = (P_x\,dx + P_y\,dy + P_z\,dz) \wedge dx = P_y\,dy \wedge dx + P_z\,dz \wedge dx$, since $dx \wedge dx = 0$. Doing the same for $Q$ and $R$ and collecting,

$$\begin{aligned}
&P_y\,dy \wedge dx + P_z\,dz \wedge dx + Q_x\,dx \wedge dy + Q_z\,dz \wedge dy\\
&\qquad + R_x\,dx \wedge dz + R_y\,dy \wedge dz.
\end{aligned}$$

Rewrite $dy \wedge dx = -dx \wedge dy$, $dz \wedge dy = -dy \wedge dz$ and $dx \wedge dz = -dz \wedge dx$, and group the terms with the same basic 2-form. For the third line, rule (b) gives $dG_1 \wedge dy \wedge dz + dG_2 \wedge dz \wedge dx + dG_3 \wedge dx \wedge dy$. In $dG_1 \wedge dy \wedge dz$, only the $dx$ term of $dG_1$ survives, giving $\partial_x G_1\,dx \wedge dy \wedge dz$. Similarly $dG_2 \wedge dz \wedge dx = \partial_y G_2\,dy \wedge dz \wedge dx = \partial_y G_2\,dx \wedge dy \wedge dz$ (two swaps), and $dG_3 \wedge dx \wedge dy = \partial_z G_3\,dx \wedge dy \wedge dz$. □

The curl, with its three entries in their slightly odd order, is what the single rule for $d$ produces when the 2-forms are listed cyclically. The order $(R_y - Q_z,\ P_z -$

$R_x,\ Q_x - P_y)$ was never a convention somebody chose to make students suffer; it was forced.

In $\mathbb{R}^n$ the same pattern holds at both ends: $d$ on functions is the gradient, and $d$ from $(n-1)$-forms to $n$-forms is the divergence, once the $(n-1)$-forms are matched with vector fields using alternating signs. In between, $d$ on 1-forms gives the entries of the curl matrix $D\vec{F} - (D\vec{F})^T$, one for each pair $i < j$.

**Theorem 6.8.** *If $\omega$ is a $k$-form on an open set in $\mathbb{R}^n$ whose coefficients have continuous second partial derivatives, then $d(d\omega) = 0$.*

In $\mathbb{R}^3$, by Theorem 6.7, this is exactly Theorem 3.1, which says that $\operatorname{curl}(\nabla f) = \vec{0}$ and that $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$. These notes do not prove the general case, but the mechanism is the same. Each mixed partial $\partial^2 g/\partial x_i \partial x_j$ appears twice in $d(d\omega)$, once next to $dx_i \wedge dx_j$ and once next to $dx_j \wedge dx_i$. The wedge rule gives the two copies opposite signs, and equality of mixed partials makes them cancel. Symmetric meets antisymmetric, and the result is zero.

One consequence is a convenient rule for 1-forms written as $f\,dg$: if $g$ has continuous second partials, then

$$d(f\,dg) = df \wedge dg.$$

Indeed, writing $dg = \sum_j g_{x_j}\,dx_j$, rule (b) and the product rule give $d(f\,dg) = \sum_j \big(g_{x_j}\,df + f\,dg_{x_j}\big) \wedge dx_j = df \wedge dg + f\,d(dg)$, and $d(dg) = 0$.

**Exercise 6.9.** Compute $d(x^2y\,dx + xy\,dy)$ on $\mathbb{R}^2$ from the rules, and check that the coefficient of $dx \wedge dy$ is the scalar curl of $(x^2y,\ xy)$.

## 6.4 Integrals and the one theorem

Forms are made to be integrated, and the integral of a form is computed by substituting the parametrization into it.

- A 0-form $f$ is "integrated" over a finite set of points with signs by adding up $\pm f$. The boundary of a curve from $\vec{p}$ to $\vec{q}$ is $\vec{q}$ with $+$ and $\vec{p}$ with $-$.
- A 1-form is integrated along a curve by substituting $dx = x'(t)\,dt$, and so on; this is the line integral.

- A 2-form $g\,dx \wedge dy$ on a region $D$ of the plane gives $\iint_D g\,dA$. On a parametrized surface $\vec{r}(u,v)$, substituting gives small determinants, as in Example 6.4: for instance

  $$dy \wedge dz = (y_u\,du + y_v\,dv) \wedge (z_u\,du + z_v\,dv) = (y_u z_v - y_v z_u)\,du \wedge dv,$$

  and the three determinants are the entries of $\vec{r}_u \times \vec{r}_v$. So integrating $G_1\,dy \wedge dz + G_2\,dz \wedge dx + G_3\,dx \wedge dy$ over the surface gives exactly the flux $\iint_D \vec{G} \cdot (\vec{r}_u \times \vec{r}_v)\,dA$ of Definition 4.1. The cross product in the flux formula, which was justified with a picture, falls out of the wedge rule.

- A 3-form $g\,dx \wedge dy \wedge dz$ on a solid $E$ gives $\iiint_E g\,dV$.

With this dictionary, every theorem of Chapter 5 has the same shape.

| $M$ | $\omega$ | $\int_M d\omega = \int_{\partial M} \omega$ says | name |
|---|---|---|---|
| interval | function $f$ | $\int_a^b f' = f(b) - f(a)$ | FTC |
| curve | function $f$ | $\int_C \nabla f \cdot d\vec{r} = f(\vec{q}) - f(\vec{p})$ | gradient |
| plane region | 1-form | $\iint_D (Q_x - P_y)\,dA = \oint \vec{F} \cdot d\vec{r}$ | Green |
| surface | 1-form | $\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\,dA = \oint \vec{F} \cdot d\vec{r}$ | Stokes |
| solid | 2-form | $\iiint_E \operatorname{div} \vec{G}\,dV = \iint \vec{G} \cdot \vec{n}\,dA$ | divergence |

**Theorem 6.10** (Stokes's theorem, schematically). *For an oriented $k$-dimensional region $M$ with oriented boundary $\partial M$, and a $(k-1)$-form $\omega$ with continuous partial derivatives on an open set containing $M$,*

$$\int_M d\omega = \int_{\partial M} \omega.$$

"Schematically" means that this is a pattern rather than a theorem with honest hypotheses: it does not say which regions are allowed. The five theorems of Chapter 5 are its precise instances. And the pairing of $d(d\omega) = 0$ with "the boundary of a boundary is empty" in Remark 5.11 is the one formula applied twice:

$$\int_M d(d\omega) = \int_{\partial M} d\omega = \int_{\partial(\partial M)} \omega.$$

Five theorems, two zeros and one formula. Not bad for a zoo.

Math 1610 · Fall 2026

torsorlab

https://torsor.org