Initial background

language of schemes, particularly divisors — Chow

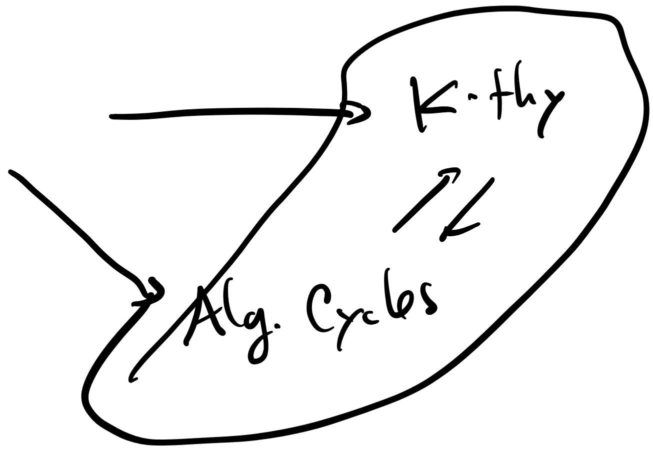
$\updownarrow$

line bundles — K-thy

$\updownarrow$

invertible sheaves

Rich / incomprehensible world of schemes



OH: Tues 5pm DRL 3E6

Algebraic cycles

Def An alg. cycle is a formal expression

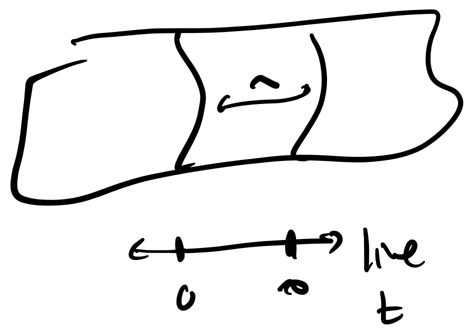
$\sum n_i [Z_i]$

$Z_i \subset X$ scheme

closed integral subscheme.
(irred closed sets)

$Z(X)$ = assoc. free ab. gp.

motivation:



gives an equiv rel and $CH(X) = \dfrac{Z(X)}{\sim}$

$\sim \leftrightarrow B(X)$

$Z(X)$ & $CH(X)$ are graded gps

(X integral Noeth scheme)

$Z^i(X)$ - gen by codim i subsets

$Z_i(X)$ - - - - dim i - -

$CH^i(X)$ $\quad CH_i(X)$ $\quad$ (Chevalley '58)

$\bigoplus CH^i(X) = CH(X)$

for X nice (e.g. smooth variety / field)

CH(X) is a ring where product $\leftrightarrow$ intersections w/ multiplicity

Contains very rich information both geometric & arithmetic

reflect (partially) properties of (singular) cohomology
or homology

Historically: Chow gps have been of primary importance in enumerative geometry

ex: complex lines in $\mathbb{CP}^3$

$\exists$ exactly 2 lines incident to 4 other general lines.

Sketch: space of these lines $\mathbb{G}(1,3)$
$$= G(2,4)$$
$$\cup$$

each of the 4 gen lines defines a subscheme $Z_1, \dots, Z_4$

$$Z_i = \{ l \in \mathbb{G}(1,3) \mid l \text{ hits line } i \}$$

$$[Z_1][Z_2]\cdots[Z_4] = 2\cdot[pt]$$

Higher algebraic cycles $\longleftrightarrow$ Motivic Cohomology

Chow gps lack various characteristic/important constructions that singular cohom has.

These deficiencies are addressed by "Higher chow gps" or "Motivic Cohomology" (Voevodsky 2000)

$$CH^i(X) \cong H^{2i,i}(X,\mathbb{Z})$$

X variety / field

$$X \longrightarrow H^n(X)$$

$$X \longrightarrow H^{p,q}(X)$$

$$X \xrightarrow{\text{étale cohom}} H^n(X_{et}, \mathbb{Z}/\ell^m)$$

$$X \xrightarrow{\text{crystalline} \ldots} H^n(X_{et}, \mathbb{Z}_\ell) \otimes H^n(X_{et}, \mathbb{Q}_\ell) \quad H^n(X_{et}, \mathbb{Q}_\ell(m))$$

$$X \longrightarrow H^{p,q}_{\mathcal{M}}(X)$$

Top

$$\begin{tikzcd}
\text{Mfld} \arrow[r] \arrow[d, squiggly] & \text{Singular Chain Complex} \arrow[dl] \\
\text{Sing Coh} &
\end{tikzcd}$$

Alg

$$\begin{tikzcd}
 & \text{CH-Mot} \arrow[d] \\
\text{Variety} \arrow[ur] \arrow[r] \arrow[dr] \arrow[dd, squiggly] & \text{Mot} \arrow[d] \\
 & \text{DMot} \arrow[dl] \\
\text{Motivic Coh} &
\end{tikzcd}$$

$CH(X)$ captured thys in all dimensions, lots of complexity, but just a slice of a "higher thy"

K-Theory

Origin: study of vector bundles (locally free sheaves)

given X $\rightsquigarrow$ construct a monoid whose elements are equiv classes of locally free sheaves

indeed by $\quad 0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$

$$\Rightarrow [\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$$

(Analogously: if R a ring — same def w/ projective modules)

$+ = \oplus$

$K^0(X)$ = "group completion" of this monoid

$[\mathcal{E}] - [\mathcal{F}]$

(or $K^0(R)$)

Further, get a ring structure induced from $\otimes$.

$K(X)$ = $K^0(X)$ has a rich structure, contains lots of information in "all dimensions"

($K_0(X)$ – coherent sheaves – not a ring in general)
$\overset{\shortparallel}{G(X)}$ analogous to homology

$K^0(X)$ comes w/ (various) filtrations

$T^i K^0(X)$ $\rightsquigarrow$ assoc. graded parts

$$CH^i(X) \leftarrow gr^i_T K^0(X)$$

$$\text{graded } K^0 \xleftrightarrow{\sim} CH^i \quad \otimes \mathbb{Q}$$

"Generalized Riemann-Roch"

Higher K-theory

K_0 K_1 K_2

$\downarrow$ $\downarrow$ $\downarrow$

schemes rings nice rings

Quillen ~'69 or '70 $K_n(X)$

Higher top K-theory Atiyah-Hirzebruch '60

$$K_0(X)_{\mathbb{Q}} \cong \bigoplus H^{2i}(X,\mathbb{Q})$$

$$K_1(X)_{\mathbb{Q}} \cong \bigoplus H^{2i+1}(X,\mathbb{Q})$$

eventually the alg version of the AH Sp. Seq. was constructed (Bloch-Lichtenbaum, Friedlander Suslin, Levine)

Goals

- Generalized Riemann-Roch
- Outline of pf of Merkurjev-Suslin thm

Intermediate goals

- Intro to chow
- Intro to K
- Taste of motivic / algebraic cobordism thy