$\mathcal{C} \rightsquigarrow Q\mathcal{C} \rightsquigarrow NQ\mathcal{C} \rightsquigarrow |NQ\mathcal{C}|$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{"} \; BQ\mathcal{C}$

exact category

additive cat

$$K_n(\mathcal{C}) = \pi_{n+1}(BQ\mathcal{C})$$

Q: what does this have to do with anything we've been talking about?

"A": we'll say something about $K_0(\mathcal{C})$ vs. $\pi_1(BQ\mathcal{C})$

<u>Next</u>: methods to work w/ $K_n(\mathcal{C})$'s.

<u>main method</u>: localization / devissage

$\rightsquigarrow$ Brown-Gersten-Quillen S.S.

we'll show (under mild assumptions - X regular var/field)

consider presheaf on X (Zariski top)

$$\mathcal{K}_n : U \rightsquigarrow K_n(U)$$

Bloch's formula.

$$H^n_{zar}(X, \mathcal{K}_n) = CH^n(X)$$

get relation of $CH^n(X)$ & subquotients of $K_0(X)$

<u>today</u>: localization (dévissage)

"<u>Recall</u>" $\mathcal{C}$ Abelian category $\mathcal{B} \subset \mathcal{C}$ full Ab. subcategory, we say $\mathcal{B}$ is a <u>Serre subcat</u> (thick) if it's closed under subobjects, quotients & extensions.

i.e. • $a \in \mathcal{B}$, $b \hookrightarrow a$ subobject (monic)
$\Rightarrow b \in \mathcal{B}$

• $a \in \mathcal{B}$, $a \twoheadrightarrow b$ epic $\Rightarrow b \in \mathcal{B}$

• $a, c \in \mathcal{B}$, $0 \to a \to b \to c \to 0$ exact in $\mathcal{C}$
$\Rightarrow b \in \mathcal{B}$.

in this case, can define a quotient category $\mathcal{C}/\mathcal{B}$ with a functor

$$\mathcal{C} \to \mathcal{C}/\mathcal{B}$$

Concretely. $\mathrm{ob}(\mathcal{C}/\mathcal{B}) = \mathrm{ob}(\mathcal{C})$

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(a,b) = \mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(a',b)$$

$= \mathrm{Hom}(a,b')$

$a' \hookrightarrow a$ such that $a/a' \in \mathcal{B}$

$b \twoheadrightarrow b'$ s.t. $\ker \in \mathcal{B}$

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(a,b) = \varinjlim_{\substack{a',b' \\ \text{as above}}} \mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(a',b')$$

ex: $\mathcal{C} = \overset{\text{f.g.}}{R}\text{-mods}$ R comm. ring.

$f \in R$. $\mathcal{B}$ = subcat of R-mods killed by some power of f.

"$\mathrm{Tors}(f)$"

$0 \to M' \to M \to M'' \to 0$

if $\exists n$ s.t. $f^n M'' = 0$

$f^n M \subset M'$

if $\exists m$ s.t. $f^m M' = 0$

$f^{n+m} M = 0$

Claim:

$$\begin{tikzcd} R\text{-mod}/\mathrm{Tors}(f) \arrow[r, "\sim", "\text{eq.}"'] & R_f\text{-mods} \\ & R\text{-mod} \arrow[ul] \arrow[u] \end{tikzcd}$$

M, N

$$\begin{tikzcd} R\text{-mod} \arrow[rr] \arrow[dr] & & R_f\text{-mod} \\ & R\text{-mod}/\mathrm{Tors}(f) \arrow[ur] & \end{tikzcd}$$

$\mathrm{Hom}_{(-)}(M,N) \overset{?}{\Rightarrow} \mathrm{Hom}_{R_f}(M_f, N_f)$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$$

$$\mathrm{Hom}_R(M', N') \to \mathrm{Hom}_R(M'_f, N'_f)$$

$$0 \to \underset{f\text{-tors.}}{N''} \to N \to N' \to 0 \qquad\qquad 0 \to M' \to M \to \underset{f\text{-tors}}{M/M'} \to 0$$

$$N_f \simeq N'_f \qquad\qquad 0 \to M'_f \xrightarrow{\sim} M_f \to \underset{0}{(\ )} \to 0$$

if $\mathcal{C}$, $\overline{\mathcal{C}}$ Ab. cats, $\mathcal{B} \subset \mathcal{C}$ Serre

an $F: \mathcal{C} \to \overline{\mathcal{C}}$ additive s.t.

$F(b) = 0$ for all $b \in \mathcal{B}$

then $\exists !$ functor

$\mathcal{C}/\mathcal{B} \to \overline{\mathcal{C}}$ s.t.

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \overline{\mathcal{C}} \\ & \mathcal{C}/\mathcal{B} \arrow[ur] & \end{tikzcd}$$

More generally, if X scheme, $U \subset X$ open

$Z = X \setminus U$ then $\mathcal{I}_Z$ ideal shf in $\mathcal{O}_X$

an $\mathcal{O}_X$-mod M is Z-torsion if M_x is $(\mathcal{I}_Z)_x$

torsion all $x \in X$ $\quad M_x/\mathcal{O}_{X,x}$.

and then $\mathcal{O}_X\text{-Mod}\Big/_{Z \text{ trs}} \simeq \mathcal{O}_U\text{-mod}$
(Coh)

<u>Note</u>: if M an $\mathcal{O}_X$-mod

$$\operatorname{supp} M = \{x \in X \mid M_x \neq 0\}$$

$\operatorname{ann} M$ ideal sheaf in $\mathcal{O}_X$

defined as $\operatorname{ann}(M)(U) = \{s \in \mathcal{O}_X \mid sM = 0\}$

$$\operatorname{supp} M = Z(\operatorname{ann}(M))$$

set theoretic

$X = \mathbb{A}^1_{\mathbb{C}}$ $\quad \mathbb{C}[x]$ $\quad M = \mathbb{C}[x]/x^2$

$\operatorname{supp} M = (x)$ $\quad M \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = 0$

$M \otimes_{\mathbb{C}[x]} \mathbb{C}[x]_{(x-a)} = 0$ $\quad x \in (\mathbb{C}[x]_{(x-a)})^*$

$\operatorname{ann}(M) = (x^2)$

$$\mathcal{O}_X\text{-mod}\Big/_{\mathcal{O}_X \text{, supp on } Z} \simeq \mathcal{O}_{X \setminus Z}\text{-mod}$$

$$0 \to \mathcal{M}_Z \to \mathcal{O}_X\text{-mod} \to \mathcal{O}_U\text{-mod} \to 0$$

where $\mathcal{M}_Z = \mathcal{O}_X$, supp on Z

We'll show/state:

get a map $BQ(\mathcal{O}_X\text{-mod}) \to BQ(\mathcal{O}_U\text{-mod})$

w/ homotopy fiber $BQ(\mathcal{M}_Z)$

Next time/next week:

"Devissage": $\Rightarrow BQ(\mathcal{M}_Z)$ hom. to $BQ(\mathcal{O}_Z\text{-mod})$

Main tools we'll make use of

Localization:

If $\mathcal{C}$ an Ab. cat, $\mathcal{B} \subset \mathcal{C}$ Serre subcat

then $BQ\mathcal{C} \longrightarrow BQ(\mathcal{C}/\mathcal{B})$ is a hom. fibration

hom. fiber $BQ\mathcal{B}$.

Resolution:

If $\mathcal{C}, \mathcal{C}'$ exact cats $\mathcal{C} \subset \mathcal{C}'$ full subcat

s.t. closed under extensions and suppose

for each $c' \in \mathcal{C}'$ $\exists$ a resolution

$$0 \to c_n \to c_{n-1} \to \cdots \to c_0 \to c' \to 0$$

$\in \mathcal{C}$

then $BQC \to BQC'$ hom. equivalence.

nes $\Longleftrightarrow$ $c_{i+1} \to c_i$ fact as $c_{i+1} \overset{\text{adm. epi}}{\twoheadrightarrow} d_i \overset{\text{adm. mono}}{\rightarrowtail} c_i$

& ex. s. in ambient ab cat

equiv. (somewhat better)

$$\begin{tikzcd}
 & & d_{i+1} \arrow[dr] & \\
c_{i+1} \arrow[r] \arrow[dr] & c_i \arrow[ur] \arrow[rr] & & c_{i-1} \\
 & d_i \arrow[u] & &
\end{tikzcd}$$

exact

? (q.i. proj.) ← prob. don't need

Cor: if X is regular variety then

$$K_n(\mathrm{Coh}(X)) \simeq K_n(\mathrm{locfree}(X))$$

Dévissage if $B \subset A$ full subcat of Ab. cat (Ab)
closed under subobjects, quotients, finite products.
such that $\forall M \in A$, $\exists$ finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \quad \text{with} \quad M_i/M_{i-1} \in B$$

$\Rightarrow BQB \to BQA$

hom equiv.

<u>Cor</u>: $Z \hookrightarrow X$ $\mathcal{M}_Z = \mathcal{O}_Z$-torsion coh sheaves

$(\mathcal{O}_Z\text{-mod}) \to \mathcal{M}_Z$

$\Rightarrow BQ(\mathcal{O}_Z\text{mod}) \xrightarrow[\text{eq.}]{\sim} BQ(\mathcal{M}_Z)$

e.g. if M f.g. I^n torsion mod over R

$(I\text{-power torsion mods}) \supset (R/I \text{ mod})$

$0 = I^n M \subset I^{n-1} M \subset \cdots \subset IM \subset M \subset 0$

$I^i M / I^{i+1} M$ I-torsion.