

$C \rightsquigarrow QC \rightsquigarrow NQC \rightsquigarrow INQC$   
 " "  
 BQC  
 exact category      additive cat

$$K_n(C) = \pi_{n+1}(BQC)$$

Q: what does this have to do with anything we've  
 been talking about?

[ "A": will say something about  $K_0(C)$  vs.  $\pi_1(BQC)$

Next: methods to work w/  $K_n(C)$ 's.  
main method: localization / devissage

Brown-Costen-Gulden S.S.

will show (under mild assumptions - X regular var/field)  
 consider question on X (Zariski top)

$$K_n: U \rightsquigarrow K_n(U)$$

Bloch's formula

$$H^n_{\text{zar}}(X, K_n) = CH^n(X)$$

get relation f  $CH^n(X)$  &  
 subquotients of  $K_0(X)$

today: localization (denisse)

"Recall"  $\mathcal{C}$  Abelian category  $B \subset \mathcal{C}$  full  
Ab. subcategory, we say  $B$  is a severe subcat  
(thick) if its closed under subobjects, quotients  
& extensions.

i.e. •  $a \in B$ ,  $b \hookrightarrow a$  subobject (monic)

$$\Rightarrow b \in B$$

•  $a \in B$ ,  $a \twoheadrightarrow b$  epic  $\Rightarrow b \in B$

•  $a, c \in B$ , or  $a \rightarrow b \rightarrow c \rightarrow 0$  exact in  $\mathcal{C}$   
 $\Rightarrow b \in B$ .

in this case, can define a quotient category

$\mathcal{C}/B$  with a functor  
 $\mathcal{C} \rightarrow \mathcal{C}/B$

Concretely.  $\text{ob}(\mathcal{C}/B) = \text{ob}(\mathcal{C})$

$$\text{Hom}(a, b) = \text{Hom}_{\mathcal{C}/B}(a, b)$$

$$\text{Hom}(a, b) \stackrel{\mathcal{C}/B}{\sim} a' \hookrightarrow a \text{ such that } a/a' \in B$$
$$b \twoheadrightarrow b' \text{ s.t. } b' \in B$$

$$\text{Hom}_{C/B}(a, b) = \varinjlim_{\substack{a' \rightarrow a \\ b' \rightarrow b \\ \text{as above}}} \text{Hom}_{C/B}(a', b')$$

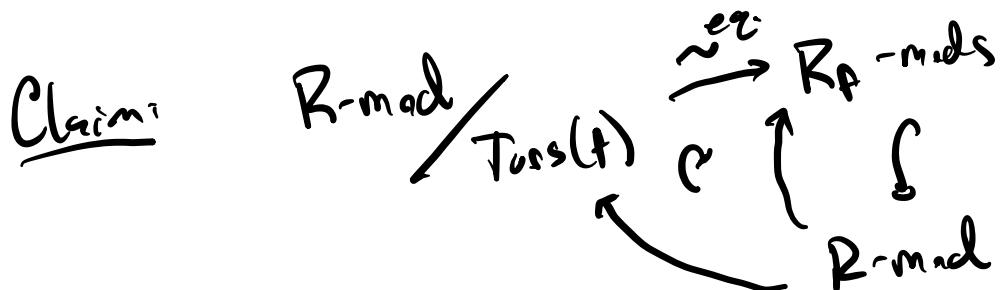
Ex:  $C = R\text{-mads}$   $R$  comm. rg.  
 $f \in R$ .  $B = \text{subcat of } R\text{-mads}$   
 $\text{Tors}(f)$  killed by some power.

 $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$   
if  $\exists n$  s.t.  $f^n M'' = 0$ 

$$f^n M \subset M'$$

$$\text{if } \exists m \text{ s.t. } f^m M' = 0$$

$$f^{n+m} M = 0$$



$M, N$

$$R\text{-mod} \xrightarrow{\sim} R_f\text{-mod}$$

$$R\text{-mod} \xrightarrow{\sim} R\text{-mod} / \text{Tors}(f)$$

$$\text{Hom}_{(R)}(M, N) \xrightarrow{\sim} \text{Hom}_{R_f}(M_f, N_f)$$

$$\text{Hom}_R(M', N') \xrightarrow{\quad} \text{Hom}_R^T(M'_f, N'_f)$$

$$\begin{array}{ccc} 0 \rightarrow & N'' \rightarrow N \rightarrow N' \rightarrow 0 & M' \rightarrow M \rightarrow M/M \rightarrow 0 \\ & f \text{-frss.} & f \text{-frss} \\ & N_f \cong N'_f & 0 \rightarrow M'_f \xrightarrow{\sim} M_f \rightarrow 0 \end{array}$$

if  $C, \bar{C}$  Ab. cats,  $B \subset C$  sene

an  $F: C \rightarrow \bar{C}$  additive s.t.

$$F(b) = 0 \text{ for all } b \in B$$

then  $F!$  funkt

$$C/B \rightarrow \bar{C} \text{ s.t.}$$

$$\begin{array}{ccc} C & \xrightarrow{\quad} & \bar{C} \\ & \searrow & \\ & C/B & \end{array}$$

More generally, if  $X$  schwe,  $U \subset X$  open

$Z = X \setminus U$  then  $\mathcal{O}_Z$  ideal sheaf in  $\mathcal{O}_X$   
an  $\mathcal{O}_X$ -mod  $M$  is  $Z$ -frssn &  $M_x$  is  $(\mathcal{O}_Z)_x$

frssn all  $x \in X$   $M_x/\mathcal{O}_{X,x}$ .

and then  $\mathcal{O}_X\text{-Mod}/\mathcal{Z}^{\text{trs}} \cong \mathcal{O}_U\text{-mod}$

(Coh)

Mate: if  $M$  an  $\mathcal{O}_X\text{-mod}$

$$\text{supp } M = \{x \in X \mid M_x \neq 0\}$$

$\text{ann } M$  ideal sheaf in  $\mathcal{O}_X$

defined as  $\text{ann}(M)(U) = \{s \in \mathcal{O}_X \mid sM = 0\}$

$$\text{supp } M = \underbrace{\mathcal{Z}(\text{ann}(M))}_{\text{set theoretic}}$$

$$X = \mathbb{A}'_{\mathbb{C}} \quad \mathbb{C}[x] \quad M = \mathbb{C}[x]/x^2$$

$$\text{s.supp. } M = (x)$$

$$M \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = 0$$

$$\text{ann}(M) = (x^2)$$

$$M \otimes_{\mathbb{C}[x]} \mathbb{C}[\sum_{(x-a)} x] \stackrel{x \in (\mathbb{C}[x])^{*}}{\sim} 0$$

$$\mathcal{O}_X\text{-mod}/\mathcal{O}_{X, \text{supp ann } Z} \cong \mathcal{O}_{X \setminus Z}\text{-mod}$$

$$0 \rightarrow \mathcal{M}_Z \xrightarrow{\text{''}} \mathcal{O}_X\text{-mod} \rightarrow \mathcal{O}_U\text{-mod} \rightarrow 0$$

We'll show/state:

get a map  $BQ(\mathcal{O}_x\text{-mod}) \rightarrow BQ(\mathcal{O}_x\text{-mod})$   
and homotopy fibr  $BQ(M_2)$

Next ingredient:

"Devisage":  $\Rightarrow BQ(M_2)$  hom. to  
 $BQ(\mathcal{O}_2\text{-mod})$

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Main tools we'll make use of

Localization

If  $C$  an Ab. cat,  $BCC$  Serre subcat  
then  $BQC \rightarrow BQ(C/B)$  is a hom.  
fibration  
hom. fibr  $BQB$ .

Resolution

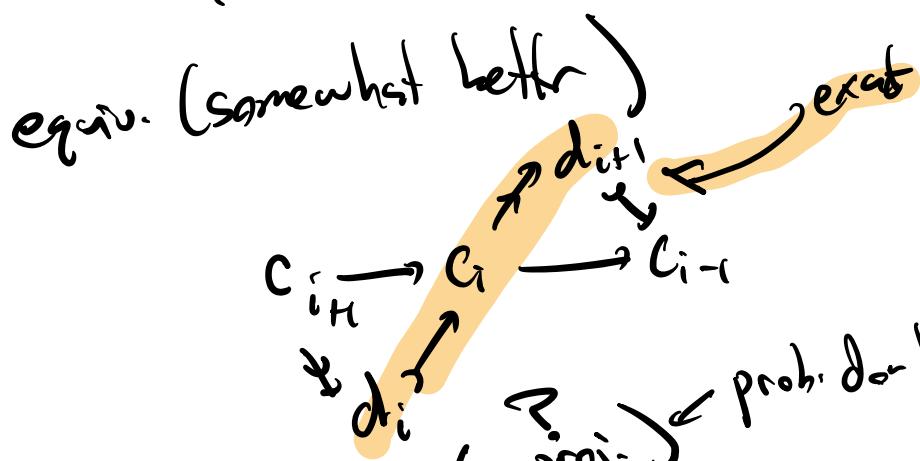
If  $C, C'$  exact cats  $C \subset C'$  full subcat  
s.t. closed under extensions and supp'  
for each  $c' \in C'$   $\exists$  a resolution

$$0 \rightarrow c_n \xrightarrow{\quad} c_{n-1} \rightarrow \dots \rightarrow c_0 \rightarrow c' \rightarrow 0$$

then  $BQC \rightarrow BQC$  hom. equivalence.

nes  $\Leftrightarrow c_{i+1} \rightarrow c_i$  fact as  $c_{i+1} \xrightarrow{\text{ad.}} d_i \xrightarrow{\text{ad.}} c_i$

& nes in ambient ab.cat



Con: if  $X$  is regular variety then

$$K_n(\text{Coh}(X)) \cong K_n(\text{locfree}(X))$$

Dérivateur if  $B \subset A$  full  $A^b$ -subcat. of  $A$ -cat

closed under subobjects, quotients, finite products.

such that  $\forall M \in A$ ,  $\exists$  finite filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M \quad \text{with}$$

$$M_i/M_{i-1} \in B$$

$$\Rightarrow BQB \rightarrow BQA$$

non equiv.

Cor:  $Z \hookrightarrow X$   $M_Z = \Omega_Z^{\text{torsion coh}}$   
shres

$$(\Omega_Z^{\text{mod}}) \rightarrow M_Z$$

$$\Rightarrow BQ(\Omega_Z^{\text{mod}}) \xrightarrow{\cong} BQ(M_Z)$$

e.g. if  $M$  f.g.  $I^n$  torsion mod over  $R$   
 $(I\text{-power torsion mod}) \supset (R/I^{\text{mod}})$

$$0 = I^n M \subset I^{n-1} M \subset \dots \subset I^0 M \subset M \neq 0$$

$$\frac{I^0 M}{I^{n+1} M} \quad I\text{-torsn.}$$