Projective bundle formula

A comm. ring. $\mathbb{P}^n_A$

$$K_i(\mathbb{P}^n_A) = K_i(\mathrm{LF}(\mathbb{P}^n_A)) = K_i(\mathrm{Coh}(\mathbb{P}^n_A))$$

$$\parallel$$

$$\bigoplus_{j=0}^{n-1} K_i(A)$$

In the case $i=0$ $\quad K_0(\mathbb{P}^n_A) = \bigoplus K_0(A)$

A = field $\quad K_0(\mathbb{P}^n_A) = \bigoplus_{j=0}^{n-1} \mathbb{Z}[\mathcal{O}(-j)]$

$$\rightsquigarrow \bigoplus_{j=0}^{n-1} K_i(A) \cdot [\mathcal{O}(-j)]$$

using the (unproven) fact that $K_*(X)$ are a graded ring.

Sketch of "lifting" this from $K_0(X)$ to $K_i(X)$

idea: express the individual projections

$$K_i(\mathbb{P}^n_A) \longrightarrow K_i(A) \quad \text{via "Kronecker"}$$

$$K_i(X) \longrightarrow K_i(Y)$$

can get these maps via $\alpha \in K_0(X \times Y)$

$$\beta \longmapsto \alpha(\beta)$$

$$\begin{tikzcd}
K_i(X) \arrow[rr] \arrow[dr, "\pi_1^*"'] & & K_i(Y) \\
& K_i(X \times Y) \arrow[ur, "(\pi_2)_*"'] &
\end{tikzcd}$$

$$\beta \longmapsto (\pi_1^* \beta) \cdot \alpha \longmapsto (\pi_2)_*((\pi_1^* \beta) \cdot \alpha)$$

$\alpha_1 \in K_0(X \times Y) \qquad \alpha_2 \in K_0(Y \times Z)$

$$\rightsquigarrow$$

$\alpha_2 \circ \alpha_1 \in K_0(X \times Z)$

$(\pi_{13})_*((\pi_{12})^* \alpha \cdot (\pi_{23})^* \alpha_2)$

$$\begin{tikzcd}
& K_0(X \times Y \times Z) \arrow[dl] \arrow[d] \arrow[dr] & \\
K_0(X \times Y) & K_0(X \times Z) & K_0(Y \times Z)
\end{tikzcd}$$

$\alpha_1 \qquad\qquad\qquad\qquad \alpha_2$

<u>Claim:</u> $\alpha_2(\alpha_1(\beta)) = (\alpha_2 \circ \alpha_1)(\beta)$

In particular: $K_0(X\times X)$ is a ring via $\circ$

$$K_0(X\times X) \longrightarrow \mathrm{End}(K_i(X)) \quad \text{all } i$$
$$\alpha \longmapsto (\beta \mapsto \alpha(\beta))$$

ring homomorphism

Basic tool to understand structure of $K_i(X)$ is "decomposition of the diagonal"

$f: X \to Y$ $\qquad$ $g: Y \to Z$

$\Gamma_f = \{(x, f(x))\}$ $\qquad$ $\Gamma_g = \{(y, g(y))\}$

$\Gamma_{g\circ f} = \{(x, g(f(x)))\}$ $\qquad$ $(\pi_{23}^*)\Gamma_g$

$\{(x, f(x), z)\} \cap \{(x, y, g(y))\}$

$(\pi_{12}^*)\Gamma_f$ $=$

$=$

$\{(x, f(x), g(f(x)))\}$

$\downarrow \pi_{13}$

$\{(x, f(g(x)))\}$

$$(\pi_{13})_* \left((\pi_{12})^* \Gamma_f \right) \cap \left((\pi_{23})^* \Gamma_g \right)$$

Turns out $K_0(X \times X)$ has unit $[\mathcal{O}_\Delta]$

$A \circlearrowright V = \sum V_i$

$e_i : V \to V_i$

$$\sum e_i = 1$$

<u>Def</u> A decomposition of the diagonal for X is a collection of orth. central idempotents $e_1, \dots, e_N \in K_0(X \times X)$ such that $\sum e_i = 1$

$\boxed{\text{Pierce decomposition}}$

$\beta \in K_i(X)$ $\quad 1 \cdot \beta = \beta$ $\quad K_i(X) = \bigoplus_j e_j(K_i(X))$

$1 = \sum e_i(\beta)$

$$e_j(K_i(X)) = K_i(A)$$

$K_0(X)$

$\operatorname{Spec} A \rightsquigarrow X = \mathbb{P}^n_A$ semiorthogonal decomp.

construct "projections onto $\mathcal{O}(i)$"

- K-thy of $\mathbb{A}^1$
- → Severi-Brauer varieties / Central simple algebras
- → BGQ-spectral sequence (Bloch's formula, Gersten conjecture)

concrete	abstract
	ss of exact couple
	(Rotman (old) hom alg)
	Eisenbud comm alg.
	Notes from my class

Vakil's rising sea (exercises)

Mcleary User's guide to Spectral seq's.

You could have invented ss's.

Intro to CSAlgebra, Brauer group,
Galois cohom & Norm-residue isom. thm
(Conjecture)

$i^2 = -1 = j^2 \quad ij = -ji$

$F \ni \rho$ prim. n^{th} root of 1.

$(a,b)_\rho \quad i^n = a \quad j^n = b \quad ij = \rho ji$

"symbol"

these are sometimes division algebras
($\text{iff } b \notin N_{F(\sqrt[n]{a})/F}$)

always "central simple algebra"

Def A/F is CSA if $\dim_F A < \infty$
$Z(A) = F$
A simple

Wedderburn-Artin: $A \simeq M_m(D)$ D CSA$/F$
which is a division alg
"D CDA"
D unique up to $\simeq$

$A \sim B$ $\quad$ A, B CSA$/F$

if same "underlying division alg"

$A \simeq M_m(D)$ $\quad$ $B \simeq M_n(D')$

$$D \simeq D'$$

$M_n(A) \simeq M_m(B)$ $\quad$ "Brauer equivalence"

Q: What do division algebras look like?

Related Q: What do CSA's look like up to Br equiv?

D $\qquad$ $M_m(D)$

$(a,b)_F$

$$\begin{array}{l} E \\ \mid \langle \sigma \rangle \text{ Cyclic Gal. ext} \\ F \quad \sigma^n = 1 \end{array}$$

$$(E, \sigma, b) = \bigoplus_{i=0}^{n-1} E \cdot u^i$$

$$u\lambda = \sigma(\lambda) u$$

$E = F(\sqrt[n]{a})$ $\qquad$ $u \mapsto j$ $\quad$ $u^n = b$

Main result of Albert-Brauer-Hasse-Noether

every div. alg over # field is of the form

(E, σ, b)

for some E/F cyclic.

"Cyclic Algs"

$$\begin{array}{l} E \\ \big|\, G \\ F \end{array} \qquad (E, G, c) = \bigoplus_{g \in G} E u_g$$

$$u_g \lambda = g(\lambda) u_g$$

u

$$u_g u_h = c(g,h) u_{gh}$$

$$c : G \times G \longrightarrow E^*$$

$$u_g(u_h u_k) = (u_g u_h) u_k$$

$$g(c(h,k))\, c(g,hk) = c(g,h)\, c(gh,k)$$

$$Z^2(G, F^\times) = \{ c : G\times G \to E^\times \mid \overset{\prime}{\text{holds}} \}$$

"2-cocycles"

<u>Fact:</u> if $c \in Z^2(G, E^\times)$ for $\begin{array}{c} E \\ \mid G \\ F \end{array}$

then (E, G, c) is <u>always</u> a CSA.

<u>Amazing Fact</u>: every CSA is Br. equivalent to one of these!

<u>Observation:</u> $Z^2(G, E^\times)$ is a group! (Ab.)

<u>Def</u> if A is a CSA, let $[A]$ denote the Brauer eq. class of A.

$$[A] + [B] = [A \otimes B]$$

$$[A^{op}] = -[A]$$

Group called "Brauer group" $Br(F)$

$Z^2(G, E^\times) \longrightarrow Br(F)$ hom.

Obvious way to chge a cocycle $c \to c'$

$(E, G, c) \cong (E, G, c')$

$$a : G \to E^\times \quad\quad u_g \rightsquigarrow a(g)\, u_g \rightsquigarrow v_g$$

$$da : G \times G \to E^\times$$

$$g, h \mapsto a(g)\, g(a(h))\, a(gh)^{-1} \quad\quad c'(g,h)\, v_{gh} = v_g v_h$$

$$a(g)\, u_g\, a(h)\, u_h$$

$$a(g)\, g(a(h))\, c(g,h)\, u_{gh} = a(g)\, g(a(h))\, u_g u_h$$

$$\underbrace{a(g)\, g(a(h))\, a(gh)^{-1}}_{c'(g,h)}\, c(g,h)\, \underbrace{a(gh)\, u_{gh}}_{v_{gh}}$$

$$C^i(G, E^\times) = \{G^i \to E^\times\}$$

$$C^1 \xrightarrow{d} C^2 \supset Z^2$$

define $H^2(G, E^\times) = \dfrac{Z^2(G, E^\times)}{\mathrm{im}(d)} = \dfrac{Z^2(G,E^\times)}{B^2(G, E^\times)}$

$H^2(G, E^\times) \longrightarrow \mathrm{Br}(F)$ is injective.

$$\begin{tikzcd}[column sep=small]
N, L \arrow[d, dash] \arrow[dd, dash, bend left, "G'"] & \\
E \arrow[d, dash, "G'/N = G"'] & \\
F &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z^2(G, E^\times) \arrow[d] \\
Z^2(G', L^\times)
\end{tikzcd}$$

$$\varinjlim_{\substack{E \\ G' \, F}} H^2(G, E^\times) \xrightarrow{\sim} \mathrm{Br}(F)$$

$$\mathrm{Br}(F) = H^2_c(\mathrm{Gal}(F^{sep}/F), (F^{sep})^\times) = H^2(F, \mathbb{G}_m)$$

<u>Q</u>: If $\mu_n \subset F$, is $\mathrm{Br}(F)$ generated by symbols $(a,b)_p$?

<u>Thm</u> (Merk-Suslin) <u>Yes</u>.

$K_*(A)$ $\qquad$ $K_*(SB(A))$

A/F CSA $\qquad$ $SB(A)$ "twisted forms of proj space"

$$K_*(SB(A)) \cong \bigoplus_{i=0}^{n-1} K_*(A^{\otimes i})$$