K-thy agenda : BGG sp. seq.

Alg agenda: Alg structures, alg gps, hom varieties.

The story of algebraic structures

Various (not obviously related) alg. structures
(f. dim'l vector spaces w/ extra nice structures)

have come up:

- Central simple algebras ⟷ Division algebras
  $M_n(D)$ Brauer gp

- Quadratic forms

  Def: $q \in K[x_1, \ldots, x_n]$ deg 2

  main question: is it isotropic
  ‖
  $\exists$ soln to $q(\vec{v}) = 0$?

  Q: for which $a \in K$ does $q(\vec{v}) = a$ have a solution?

290 thm (Bhargava, Hanke) all $n > 0$ $n = q(\vec{v})$ for $q \in \mathbb{Z}[x_1, \ldots, x]$ integral $\vec{v} \in \mathbb{Z}^n$

$\iff$ all n up to 290 are represented.

over fields every pos rat'l # is represented by any pos-def q. form in at least 4 variables.

over $\mathbb{Q}$

$x^2+3y^2+9z^2+11w^2$

Def q is "anisotropic" if $q(\vec{v})=0$ has no nontrivial solns.

anisotropic forms form a sp "Witt sp"

$W(K)$ $\quad q\perp q'$ $\quad +$

$\quad q\otimes q'$ $\quad \times$

- along similar lines, if L/K quadratic, can consider Hermitian forms w/r/t to L/K

$h: L\times L \longrightarrow L$ sesquilinear form

$h(x,y)=\overline{h(y,x)}$ $\quad \overline{} =$ aut of L/K

and L-linear in first coordinate.

- Octonion algebras

(a,b,c)

$i^2=a \quad j^2=b \quad k^2=c$

$ij=-ji \quad jk=-kj$

$ik=-ki$

Quat alg: $(a,b)_{-1}$

$i^2=a \quad j^2=b$

$ij=-ji$

Look up the rest of the multiplication.

nonassociative

"alternative"

unrelated variable names →

$$a(bc) - (ab)c = \{a,b,c\}$$
$$= (ba)c - b(ac)$$

- Albert Algebras

Lie. $[a,b] = ab - ba$

Jordan: $a \cdot b = \frac{ab+ba}{2}$

basic identities
- $[a,a] = 0$
- $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$

All of them:
if $\mathfrak{g}, [\,,\,]$ satisfies

then $\exists$ A assoc. and

$\mathfrak{g} \hookrightarrow A$ s.t.

commutator on A restricts to $[\,,\,]$ on $\mathfrak{g}$.

write down a few identities
- $ab = ba$
- $a^2(ba) = (a^2b)a$

Def a Jordan alg is a non ass alg w/ identities.

Special if $J \hookrightarrow A$
s.t. $a \circ b = \frac{ab+ba}{2}$ in A.

exceptional if not special

Def An Albert algebra is a 27 dim'l nonspecial (exceptional) Jordan algebra.

($\mathbb{A}^3$ and his algebra)

3x3 Hermitian Octonionic matrices

$$\begin{bmatrix} \lambda_1 & a & c \\ \bar{a} & \lambda_2 & b \\ \bar{c} & \bar{b} & \lambda_3 \end{bmatrix} \qquad a \cdot b = \frac{ab+ba}{2}$$

All these structures arise naturally in the classification of alg gps.

simple alg gps $\longleftrightarrow$ aut's of these structures.

V spaces $\longleftrightarrow$ GL_n, $\left.\begin{matrix} SL_n \\ PGL_n \end{matrix}\right\} A$

Hermitian forms $\longleftrightarrow$ A "outer"

Quad forms $\longleftrightarrow$ B, D

$O(q)$

$SO(q)$

$GO(q)$ $\quad$ $PGO(q)$

Symplectic forms $\longleftrightarrow$ C

Octonions $\longleftrightarrow G_2$

Albert $\longleftrightarrow F_4$

E_6, E_7, E_8

each of these gps
$$\begin{tikzcd} & A \\ G \arrow[ur] \arrow[r] \arrow[dr, dash] & B \\ & C \end{tikzcd}$$

has associated to it a short list of projective varieties on which they act transitively.

$A \longleftrightarrow$ Vector spaces $\longleftrightarrow GL$

$PGL \curvearrowright \mathbb{P}^n \quad Gr(k,n)$

$D, B \longleftrightarrow$ q. forms $\longleftrightarrow O_n$

$SO_n \curvearrowright$ Quadric hypersurfaces

$G \longleftarrow$ actions $\longleftrightarrow$ varieties

in all cases, geometry, Chow gps, K-thy play a large role in alg/arith props of algebraic structures

<u>Notation:</u> for a scheme X, $X^{(p)} = \{x \in X \mid \operatorname{codim} x = p\}$

Theorem X an irreducible Noeth scheme

$\exists$ spectral sequence $E_1^{p,q} = \coprod_{x \in X^{(p)}} K_{-p-q}(k(x))$ (res. field of x) $\Rightarrow G_{-p-q}(X)$

which converges.

Filtration (topological / coniveau)

$$\mathrm{Coh}(X) \supset \mathrm{Coh}(X)^p$$

$\mathrm{Coh}(X)^p$ = full subcat w/ objects coh. sheaves $\mathcal{F}$ s.t. $\mathrm{supp}(\mathcal{F})$ has codim $\geq p$.

$$K_n(\mathrm{Coh}(X)^p) \longrightarrow K_n(\mathrm{Coh}(X))$$

image $= F^p K_n(\mathrm{Coh}(X))$

$F^p G_n(X)$

Observation 1: $\mathrm{Coh}(X)^{p+1} \hookrightarrow \mathrm{Coh}(X)^p \hookrightarrow \mathrm{Coh}(X)$

Serre subcategory.

Observation 2:

$$\mathrm{Coh}(X)^{p} / \mathrm{Coh}(X)^{p+1} \simeq \coprod_{x \in X^{(p)}} \mathcal{A}(\mathcal{O}_{X,x})$$

Def: if R local ring, $\mathcal{A}(R)$ = cat. of finite length R-modules.

given Ab. cats A_λ $\lambda \in I$

define $\coprod A_\lambda$

objects = $(a_\lambda)_{\lambda \in I}$ s.t. all but finitely many are $\simeq 0$

idea of equivalence

if given a module $\mathcal{F}$ supported on $Z \subset X$ irred.

consider $\mathcal{F}_t(U) := \{ f \in \mathcal{F}(U) \mid \mathrm{supp}(f) \subsetneq Z \}$

$\mathcal{F}/\mathcal{F}_t \rightsquigarrow$ module over $\mathcal{O}_{X,\eta(Z)}$

this gives fibration sequence

$$BQ\,Coh(X)^{p} \to BQ\left(Coh(X)^{p}/Coh(X)^{p+1}\right)$$

fibr $BQ\,Coh(X)^{p+1}$

$$K_{n+1}\big(\underbrace{Coh(X)^{p}/Coh(X)^{p+1}}_{\coprod_{x\in X^{(p)}} A(\mathcal{O}_{X,x})}\big) \to K_n(Coh(X)^{p+1}) \to K_n(Coh(X)^{p}) \to \cdots$$

$$= \coprod_{x\in X^{(p)}} K_{n+1}(A(\mathcal{O}_{x,x}))$$

$$= K_{n+1}(\mathcal{O}_{x,x}/m_x) = K_{n+1}(k(x))$$

<u>observation</u>: if (R,m) local M f. length

$$0 = m^i M \subset \cdots \subset mM \subset M$$

$m^iM/m^{i+1}M$ killed by m.

subcat R/m-mods in $A(R)$

$$\text{Devissage} \Rightarrow K_n(\overset{f.g.}{R/m}) \simeq K_n(A(R))$$

get LES's:

$$K_i(Coh(X)^{p+1}) \to K_i(Coh(X)^{p}) \to \coprod_{x\in X^{(p)}} K_i(k(x)) \to K_{i-1}(Coh(X)^{p+1}) \to \cdots$$