Part 1
(BGQ etc)

Part 2
(Severi-Brauer Varieties)

Reminder of Exact couples

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

exact couple

in practice $D = \bigoplus D^{p,q}$

$\rightsquigarrow$ derived couple

$$\begin{tikzcd} \alpha D \arrow[rr, "\alpha'"] & & \alpha D \arrow[dl, "\beta'"] \\ & HE \arrow[ul, "\gamma'"] & \end{tikzcd}$$

$HE = \dfrac{\ker d}{\operatorname{im} d}$

$\alpha D = \alpha(D)$

$\beta\gamma = d : E \to E$

$\alpha' = \alpha|_{\alpha D}$

$\beta' = \beta \circ \alpha^{-1}$ $\quad \gamma' = \overline{\gamma}$

$$\begin{tikzcd} D_1 \arrow[rr] & & D_1 \arrow[dl] \\ & E_1 \arrow[ul] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} D_2 \arrow[rr] & & D_2 \arrow[dl] \\ & E_2 \arrow[ul] & \end{tikzcd}$$

$$E_n = \frac{\ker d_{n-1}}{\operatorname{im} d_{n-1}}$$

for us, we'll start w/ $E_1 = \bigoplus E_1^{p,q}$

$D_1 = \bigoplus D_1^{p,q}$

$$\begin{tikzcd} D_1 \arrow[rr, "{(1,-1)}"] & & D_1 \arrow[dl, "{(0,0)}"] \\ & E_1 \arrow[ul, "{(1,0)}"] & \end{tikzcd}$$

$$D_1^{p,q}(X) = K_{-p-q}(\mathcal{M}(X)^p)$$

$$E_1^{p,q}(X) = \bigoplus_{x \in X^{(p)}} K_{-p-q}(k(x))$$

$$\begin{tikzcd} D_1^{p+1,q-1}(X) \arrow[r] \arrow[d, equal] & D_1^{p,q}(X) = K_{-p-q}(\mathcal{M}(X)^p) \arrow[d, equal] & E_1^{p,q} \arrow[d, equal] \\ K_n(\mathcal{M}(X)^{p+1}) \arrow[r] & K_n(\mathcal{M}(X)^p) \arrow[r] & \bigoplus_{x \in X^{(p)}} K_n(k(x)) \arrow[dll] \\ K_{n-1}(\mathcal{M}(X)^{p+1}) \arrow[r] & & \end{tikzcd}$$

$n = -p-q$ $\quad \boxed{q = p-n}$

<u>in this context</u>: Set $A^n = \lim\limits_{\to q} D^{n-q,q} = K_{-n}(\mathcal{M}(X)^0)$

$= G_{-n}(X)$

.... ?

punchline: get a convergent ss. (see Srinivas Appendix C)

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K_{-p-q}(k(x)) \Longrightarrow G_{-p-q}(X)$$

filtration on $G_n(X)$ is given by

$$F^p G_n(X) = \mathrm{im}\left(K_n(\mathcal{M}(X)^p) \to \underset{\substack{\parallel \\ G_n(X)}}{K_n(\mathcal{M}(X))} \right)$$

induced by $\mathcal{M}(X)^p \longrightarrow \mathcal{M}(X)$

e.g. $F^p G_0(X)$ = gen by classes of sheaves $\mathcal{F}$ on X whose support has codim $\geq p$

Let's take a look

$$E_1^{p,q} \longrightarrow E_1^{p+1,q}$$

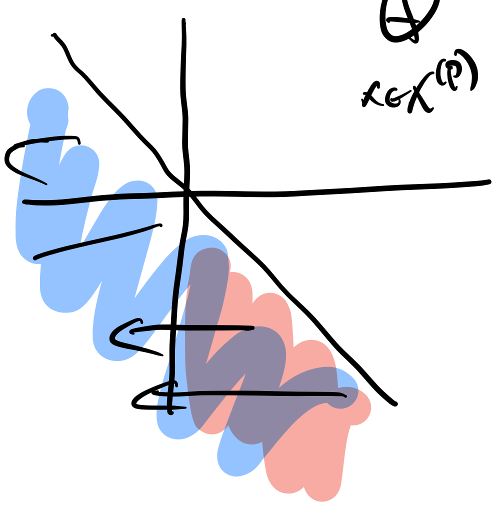
$$E_1^{0,0} \longrightarrow E_1^{1,0} \qquad 0$$

$$\bigoplus_{x \in X^{(0)}} K_0(k(x)) \qquad \bigoplus_{x \in X^{(1)}} K_{-1}(k(x))$$

$E_1^{p,q} \neq 0$ if $-p-q \geq 0$ $\quad p+q \leq 0$

$$\bigoplus_{x \in X^{(p)}} K_{-p-q}(k(x)) \qquad p \leq -q$$

$p \geq 0$



$$E_1^{0,0}$$

$$E_1^{0,-1} \longrightarrow E_1^{1,-1}$$

$$E_1^{0,-2} \longrightarrow E_1^{1,-2} \longrightarrow E_1^{2,-2}$$

"residue"

$$\begin{tikzcd}
\bigoplus_{x\in X^{(0)}} K_0(k(x)) & & \\
\bigoplus_{x\in X^{(0)}} K_1(k(x)) \arrow[r] & \bigoplus_{x\in X^{(1)}} K_0(k(x)) & \\
\bigoplus_{x\in X^{(0)}} K_2(k(x)) \arrow[r] & \bigoplus_{x\in X^{(1)}} K_1(k(x)) \arrow[r] & \bigoplus_{x\in X^{(2)}} K_0(k(x))
\end{tikzcd}$$

Weil divisors

codim 2 cycles

$$\begin{tikzcd}
\bigoplus_{x\in X^{(0)}} \mathbb{Z}[x] & & \\
\bigoplus_{x\in X^{(0)}} k(x)^* \arrow[r, "\mathrm{div}"] & \bigoplus_{x\in X^{(1)}} \mathbb{Z}[x] & \\
\bigoplus_{x\in X^{(0)}} K_2(k(x)) \arrow[r] & \bigoplus_{x\in X^{(1)}} k(x)^* \arrow[r, "\mathrm{div}"] & \bigoplus_{x\in X^{(2)}} \mathbb{Z}[x]
\end{tikzcd}$$

Right hand terms give: $E_2^{n,-n} = CH^n(X)$

$$= \operatorname{coker}\Big(\bigoplus_{x\in X^{(n-1)}} k(x)^* \to \bigoplus_{x\in X^{(n)}} \mathbb{Z}[x] \Big)$$

another interesting observation:

consider the graded pieces

$$E_\infty^{p,-p}(X) \simeq F^p G_0(X) \big/ F^{p+1} G_0(X)$$

$\shortparallel$

quotients of

$E_1^{p,-p}(X)$

$\shortparallel$

$CH^p(X)$

(fact: kernel of

$$CH^p(X) \longrightarrow \frac{F^p G_0(X)}{F^{p+1} G_0(X)}$$

is torsion.)

side observation

$$E_1^{p,q} \longrightarrow E_1^{p+1,q}$$

$$E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$$

$\vdots$

$$E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

for $p=1$, $E_2^{1,-1} = E_\infty^{1,-1} = \dfrac{F^1 G_0(X)}{F^2 G_0(X)}$

One more remark

Conjecture (Gersten) if X is Spec A, A a local regular dmd

then the augmented complex of E_1 terms

$$0 \to K_n(X) \to \bigoplus_{x \in X^{(0)}} K_n(k(x)) \to \bigoplus_{x \in X^{(1)}} K_{n-1}(k(x)) \to \cdots \bigoplus_{x \in X^{(d)}} K_{n-d}(k(x)) \to 0$$

is exact. (true for local rings of regular schemes over fields by Quillen, more generally for eq. char regular rings by Panin)

consequence of this: get a flasque resolution of the Zariski sheaf

$$\mathcal{K}_n : U \longrightarrow K_n(U)$$

$$0 \to \mathcal{K}_n \longrightarrow \mathcal{F}_{n,0} \longrightarrow \mathcal{F}_{n,1} \longrightarrow \cdots$$

$$\mathcal{G}_{n,p} = \bigoplus_{x \in X^{(p)}} i_* K_n(\kappa(x))$$

$i : \operatorname{Spec} \kappa(x) \longrightarrow X$

inclusion.

Consequence of Gersten conjecture

$$E_2^{p,q}(X) = H^p(X_{Zar}, \mathcal{K}_{-q})$$

in particular $E_2^{p,-p} = CH^p(X)$

$$\| $$

$$H^p(X_{Zar}, \mathcal{K}_p)$$

"Bloch's Formula"

K-cohomology groups

Brief digression:

Severi-Brauer Varieties

Def A central simple algebra over F is an F-alg A s.t. $A \otimes_F F^s \cong M_n(F^s)$ some n. $\boxed{n = \deg A}$ F^s = separable closure

i.e. "forms" of matrix algebras

Consider right ideals of $M_n(F)$

exercise: every right ideal $I \triangleleft_r \mathrm{End}(V)$ is of the form $\mathrm{Hom}(V, W) \subset \mathrm{End}(V)$ some $W \subset V$.

i.e. $\exists$ bijection $\underset{\text{subspaces}}{\{W < V\}} \longleftrightarrow \underset{\text{r. ideals.}}{\{I \triangleleft_r \mathrm{End}(V)\}}$

(columns in W) $\dim_F I = (\dim W)(\dim V)$

Observe: for any f.d. alg A, right ideals of dim d forms a closed subvariety of $Gr(d,A)$
$RI_d(A)$

and if A is CSA, consider $RI_n(A)$
deg $n = \sqrt{\dim_F A}$

is a subvar of $Gr(n,A) = Gr(n,n^2)$

F^s-pts: $RI_n(A)(F^s) = R_n(A \otimes_F F^s)(F^s)$

$$\updownarrow$$

$n \cdot 1 = n$-dim'l r. ideals of $A \otimes F^s \cong M_n(F^s)$

$$\updownarrow$$

1 dim'l subspaces of $(F^s)^n$

$$\updownarrow$$

$$\mathbb{P}^{n-1}_{F^s}$$

$$RI_n(A)_{F^s} \cong \mathbb{P}^{n-1}_{F^s}$$

Def $SB(A) = RI_n(A)$

"Semi-Braun" Variety

Châtelet ~1950