Algebraic cycles & Chow gps

(Fulton's Intersection thy
mostly ch 1 (also ch 5, App B.6))

From now on:
scheme = Noetherian separated scheme
(eventually finite type over a field)

"Remind"

X a scheme, a prime cycle is an irreducible closed subset of X (equivalently, correspond to scheme-theoretic pts / integral closed subschemes)

$Z(X)$ = the free Ab gp generated by prime cycles

$Z_i(X)$ = " " " " " " " " of dim i

elements of $Z(X)$ are represented as

$$\sum n_i [Z_i]$$

Plan: • Rational equivalence
• Functoriality of cycles (push forward
pullback)

• approach intersections / pullback over
closed imbedding

Rational equivalence

$R(W)$ " $F(W)$ …
"
if W is an integral scheme $K(W)$: field of rat'l functions

given $f \in K(W)^*$

in Hartshorne we define $\mathrm{div}(f)$
(in case where W is "R1C1")
regular in codim 1.

$$\mathrm{div}(f) = \sum_{\substack{D \subset W \\ \text{codim } 1 \text{ pts}}} v_D(f)\,[D]$$

"regular in codimension one"

$v_D(f)$: $\mathcal{O}_{W,D}$ if R1C1 this is a dvr
localizing at D on W
get a valuation on $\mathcal{O}_{W,D}$
v_D

$\therefore \mathrm{frac}(\mathcal{O}_{W,D}) = k(W)$

alternately, even if $\mathcal{O}_{W,D}$ not regular, for $f \in \mathcal{O}_{W,D}$

define: $v_D(f) = \mathrm{len}_{\mathcal{O}_{W,D}}(\mathcal{O}_{W,D}/f)$

in general, if $f \in \mathrm{frac}(\mathcal{O}_{W,D})$

$f = f_1/f_2$ define

$\underline{v_D(f) = v_D(f_1) - v_D(f_2)}$

app. A7

<u>Facts</u> (not prove) this gives a finite sum

$$\mathrm{div}(f) = \sum_{D \subset W} v_D(f)[D]$$

we'll see it's nicely compatible w/ prior def.

<u>Recall</u>: Hartshorne linear equivalence of divisors

$\mathrm{Div}(X) =$ formal sums of codim 1 pts

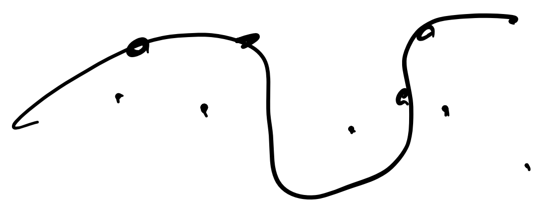
$= Z_{\dim X - 1}(X)$

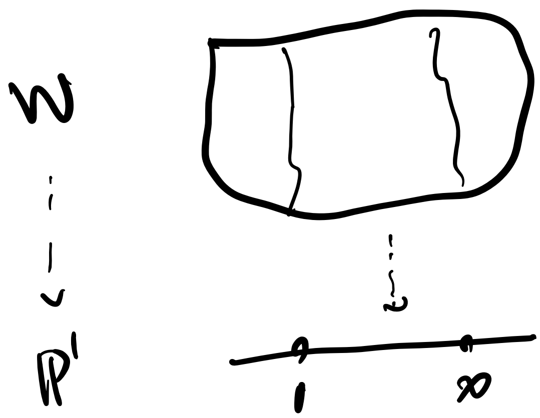
$B_{\dim X-1}(X) =$ generated by $\{(\mathrm{div}\, f) \mid f \in k(X)^{*}\}$

more generally,

$$B_i(X) = \text{gen by } \{ (\operatorname{div} f) \mid f \in k(W)^*,\ W \subset X \text{ closed integral subscheme of dim } i+1 \}$$

$Z_0(X)$





Def $CH_i(X) = Z_i(X) / B_i(X)$

$$CH(X) = \bigoplus CH_i(X)$$

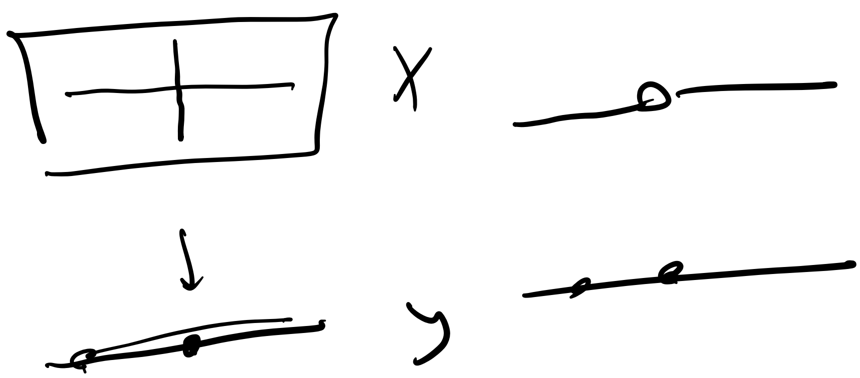
"Functoriality"

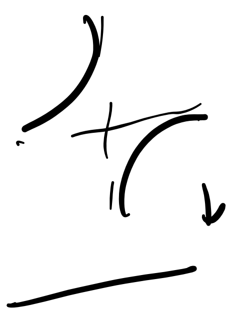
i.e. given $f: X \longrightarrow Y$ morphism

roughly we want say that for $Z \subset X$ $W \subset Y$

$f_*[Z]$ = "class of image $f(Z)$ w/ multiplicity"

$f^*[W]$ = " . . . inverse image $f^{-1}(W)$, — — "





For these reasons, we will initially restrict to

define f^* only when f is flat

and f_* only when f is proper

Def: if $f: X \to Y$ is proper then for $Z \subset X$ integral closed

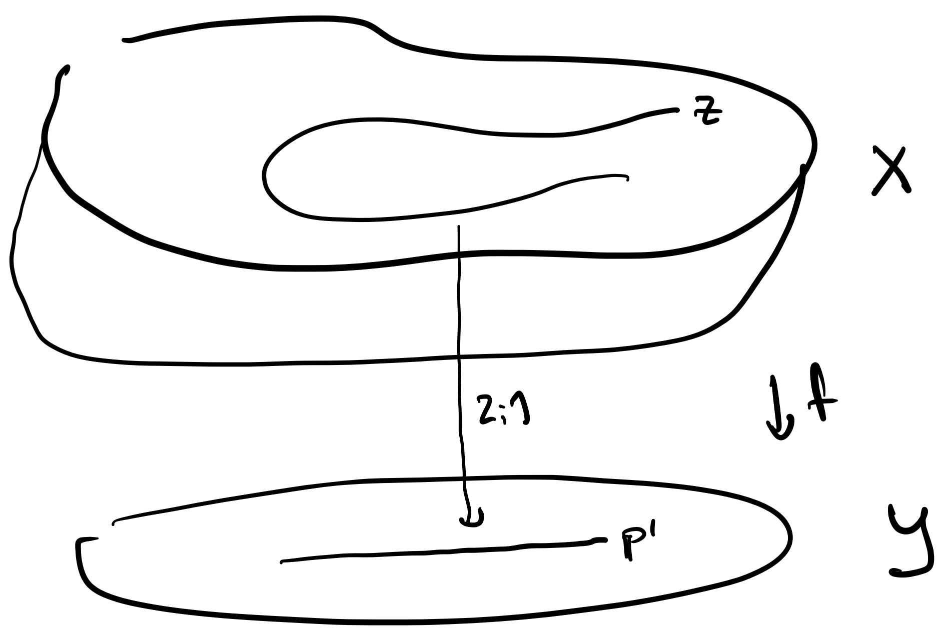
$$f_*[Z] = \begin{cases} 0 & \text{if } \dim f(Z) < \dim Z \\ [k(Z):k(f(Z))]\,[f(Z)] & \text{if } \dim f(Z) = \dim Z \end{cases}$$

generic degree of cov'g $Z \to f(Z)$

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ {\scriptstyle f|_Z}\downarrow & & \downarrow{\scriptstyle f} \\ f(Z) & \hookrightarrow & Y \end{array}$$

$k(f(Z)) \xrightarrow{f|_Z} k(Z)$ (dominant)

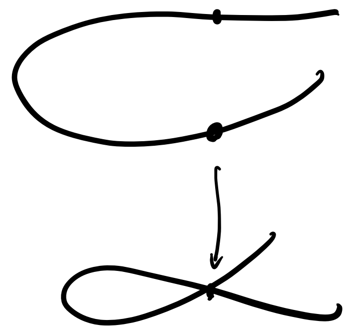
if dim's same then this is a finite field extension



$f_*[Z] = 2[\mathbb{P}^1]$

Work involved in showing this respects rat'l equivalence and therefore defines a map $CH_i(X) \xrightarrow{f_*} CH_i(Y)$

FACT: gives a well defined map on Chow gps.



Pullbacks

Def if $f: X \to Y$ flat then we define for $W \subset Y$ closed integral

$$f^*[W] = [f^{-1}W] \quad \left(= \sum_i n_i [W_i] \ldots\right)$$

problem w/ this def is here by $f^{-1}W$ we mean scheme-theoretic inv. image.

To make this make sense we need to define for $W \subset X$ closed subscheme

$[W] \in Z(X)$

if W_i are the minimal primes of W (irred comps of)

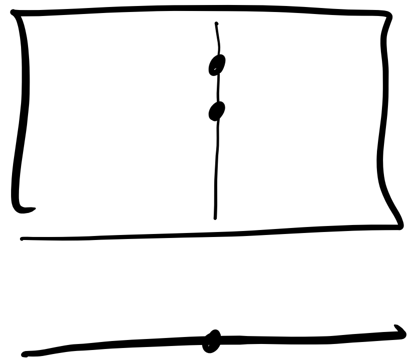
then we define $\mathrm{mult}_{W_i}(W) = \mathrm{len}(\mathcal{O}_{W,W_i})$

$$[W] \equiv \sum_{W_i} \mathrm{mult}_{W_i}(W)\,[W_i]$$

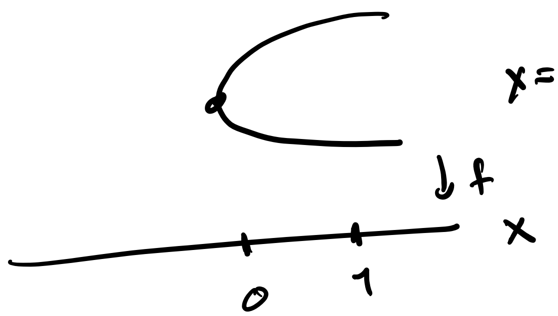
In general, if $f: X \to Y$ flat relative dim d

X, Y irred

$Z_i Y \xrightarrow{f^*} Z_{i+d}(X)$



induce maps on Chow grps $CH_i(Y) \xrightarrow{f^*} CH_{i+d}(X)$



$x = y^2$

$$\frac{k[x,y]}{x-y^2}$$
$$\uparrow$$
$$k[x]$$

$$f^*[0] = \left[\mathrm{Spec}\left(k[x,y]/(x-y^2) \otimes_{k[x]} k[x]/x\right)\right]$$

$$= \left[\mathrm{Spec}\left(k[x,y]/(x-y^2, x)\right)\right]$$

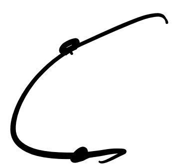
$$= \left[\mathrm{Spec}\; k[y]/y^2\right]$$

$$\mathrm{mult}_{(y)} k[y]/y^2 = \mathrm{len}\left(k[y]/(y^2)\right) = 2$$

$$f^*[0] = 2[y=0]$$

$$f^*[1] = \left[\operatorname{Spec} k[x,y]/(x-y^2,\, x-1)\right]$$

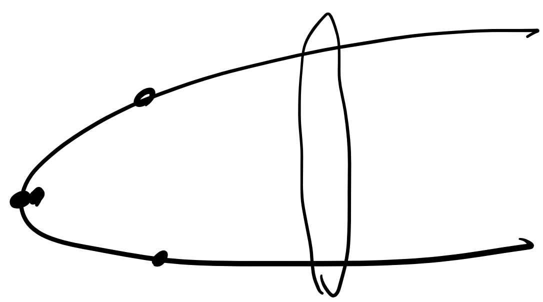
$$= \left[\operatorname{Spec} k[y]/(y^2-1)\right] \xrightarrow{\text{char} \neq 2} k[y]/(y+1) \times k[y]/(y-1)$$



$\xrightarrow{\text{char}=2}$ $k[y]/(y-1)^2$

$k = \mathbb{Q}$ pmy of -1

$k[y]/(y^2+1) = \mathbb{Q}[i]$ field.



Intersections.

idea: given $W, Z \subset X$ want to "intersect"

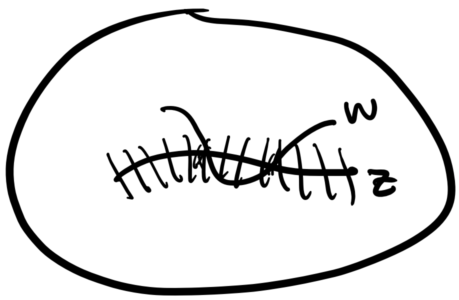
$[W]$ & $[Z]$

first approach:

consider "Gysin pullback"

$i: Z \hookrightarrow X$ $\quad$ $i^*[W]$

$[W]$ $\quad$ "intersection of Z & W inside of Z.



if Z is regularly embedded $\quad$ $N_Z X$ is a vb $/ Z$