Plan: (Almost) finish the proof of M-S theorem.

H90 for $K_2 \Rightarrow$ MS

H90 = trivality of the group $V(F)$

E/F deg p extension $E = F(\sqrt[p]{a})$ $\mu_p \subset F$

$Gal(E/F) = \langle\sigma\rangle$

$$K_2(E) \xrightarrow{\sigma-1} K_2(E) \xrightarrow{N} K_2(F)$$

homology here $\equiv V(F)$

$b \in F$

$N(\vec{x}) = b$

$SB(a,b)$

Strategy: H90 true if F is is pre-to-p closed & norm is surjective from $E^\times \to F^\times$

Can reduce to Norm surjective if we can show for $D = (a,b)_\epsilon$ ϵ p^{th} root of 1

that $V(F) \hookrightarrow V(F(SB(D)))$

$X = SB(D)$

Last time: showed $V(F) \hookrightarrow V(F(X))$

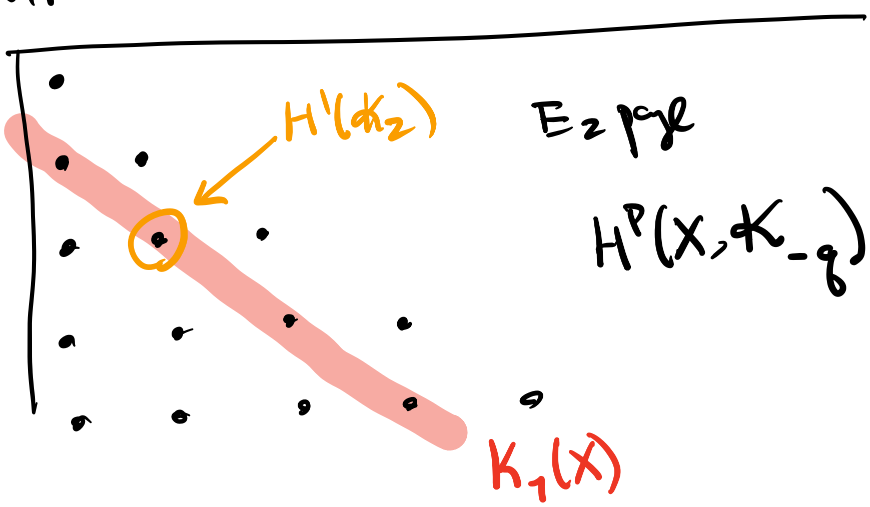
<u>if</u> $H^1(X, K_2) \hookrightarrow H^1(X_E, K_2)$

injective

<u>Goal for today:</u> $H^1(X, K_2) \hookrightarrow H^1(X_E, K_2)$

if $CH^i(X) \hookrightarrow CH^i(X_E)$.

Will use BGQ SS:



E_2 page

$$H^p(X, K_{-q}) \Rightarrow K_{-p-q}(X)$$

E_1 page

$$\bigoplus_{x \in X^{(0)}} K_*(k(x)) \Rightarrow K_* X$$

K-theory of SB varieties (in particular projective space)

D CSA deg p

<u>Theorem (Quillen)</u> $K_n(SB(D))$

$$\cong$$

$$K_n(F) \oplus K_n(D) \oplus K_n(D^{\otimes 2}) \oplus \cdots \oplus K_n(D^{\otimes p-1})$$

$$D^{\otimes i} = \underbrace{D \otimes_F D \otimes \cdots \otimes_F D}_{i \text{ times}}$$

How does this iso work?

Motivational idea: Morita theory

$N \in {}_R\mathrm{Mod}_S$ N!

$$R \qquad\qquad S$$

$$\mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(S^{op})$$

$$M \longmapsto M \otimes_R N$$

for Quillen's result want $\mathcal{O}_X - D^{\otimes i}$ bimodules

$SB(D)$

$SB(D)$
$\shortparallel$
<u>recall:</u> X parametrizes p-dim'l right ideals of D.

$\{I \triangleleft_r D \mid \dim I = p\}$ at alg. closure look like

$$D_{\bar{F}} = M_p(\bar{F}) = \mathrm{End}(V)$$

$$I = \mathrm{Hom}(V, L)$$

define $\mathcal{L}$ "tautological" $\mathcal{O}_X$-module.

$\mathcal{L}$ "at a pt $x \in X$" is $\quad x \leftrightarrow I \triangleleft D_{k(x)}$

$$\mathcal{L}_x = I$$

but these ideals are right D-modules so have an action of D (or $\mathcal{O}_X \otimes_F D$) on $\mathcal{L}$ on right.

given a left D-module M, can consider the $\mathcal{O}_X$-module $\quad \mathcal{L} \otimes_D M$

more generally, $\mathcal{L}^{\otimes i}$ is a right $D^{\otimes i}$ module

let $D^{\otimes i}$ mods $\longrightarrow$ $\mathcal{O}_X$ mods

$$M \longmapsto \mathcal{L}^{\otimes i} \otimes_{D^{\otimes i}} M$$

$$\{\text{f.g. } D^{\otimes i}\text{-mod}\} \longrightarrow \mathrm{Coh}\,\mathcal{O}_X$$

$$K_n(D^{\otimes i}) \longrightarrow K_n(X)$$

$$K_n(X) \simeq \bigoplus_{i=0}^{p-1} K_n(D^{\otimes i})$$

in case $X = \mathbb{P}^{p-1}$ $\quad K_n(\mathbb{P}^{p-1}) \simeq \bigoplus_{i=0}^{p-1} K_n(F)$

$$\begin{array}{ccc} \bigoplus^{p-1} K_n(F) & \longrightarrow & K_n(\mathbb{P}^{p-1}) \\ (a_0, a_1, \dots, a_{p-1}) & \longmapsto & \sum \underbrace{(\gamma-1)^i}_{K_0(\mathbb{P}^{p-1})} \cdot \underbrace{a_i}_{K_n(F)} \end{array}$$

$$\begin{array}{ccc} \mathbb{P}^{p-1} & & \\ \downarrow & & \\ \mathrm{Spec}\, F & & \end{array}$$

$\gamma = [\mathcal{O}(-1)]$ in $K_0(\mathbb{P}^{p-1})$

note: $\gamma - 1 = -$ hyperplane.

$$\begin{array}{ccccccccc} & & \gamma & & 1 & & [H] & & \\ 0 & \to & \mathcal{O}_{\mathbb{P}^{p-1}}(-1) & \to & \mathcal{O}_{\mathbb{P}^{p-1}} & \to & i_* \mathcal{O}_{\mathbb{P}^{p-2}} & \to & 0 \\ & & & & & & \parallel & & \\ & & & & & & \mathcal{O}_H & & \end{array}$$

$$0 \to S \xrightarrow{\cdot x} S \to S/x \to 0$$

Can show:

$$F^i K_1(\mathbb{P}^{p-1}) = \bigoplus_{j=i}^{p-1} (\gamma-1)^j K_1(F)$$

$$gr^i K_1(\mathbb{P}^{p-1}) \cong K_1(F) = F^*$$

$$F^i(K_1(X_E)) = \bigoplus_{j=i}^{p-1} (\gamma-1)^j \underbrace{K_1(E)}_{E^*}$$

$$gr^i(K_1(X_E)) = E^*$$

$$K_1(X) = \bigoplus K_1(D^{\otimes i}) \overset{\text{Platonov}}{=} \bigoplus \mathrm{Nrd}(D^{\otimes i})$$

$A = D^{\otimes i}$ CSA $\quad = F^* \oplus \mathrm{Nrd}(D^*)^{\oplus p-1}$

$$K_1(A) = A^* / [A^*, A^*]$$

$$1 \to SK_1(A) \longrightarrow K_1(A) \longrightarrow F^* \longrightarrow 1$$

$$a \longmapsto \mathrm{Nrd}(a)$$

Platonov: $SK_1(A) = 0$ if ind A is prime.

$\mathrm{Nrd}: K_1(A) \longrightarrow F^*$ injective

$K_1(A) = \mathrm{Nrd}(A^*)$

Why is M-S interesting / important?

$K_2(F)$ $\qquad$ $H^2(F, \mu_\ell^{\otimes 2})$

$K_n^M(F)/\ell \xrightarrow{\sim} H^n(F, \mu_\ell^{\otimes n})$ from e.g.

$H^1(F, \mu_\ell) = F^\times/(F^\times)^\ell$

$H^n(F, \mu_\ell^{\otimes n-1})$ natural recipient of invariants.

$K_n(X)$ Remarks: E_1 page $\qquad \partial_1 =$ "gen. div of poles"

$$\bigcup \Big| K_n(k(X)) \xrightarrow{\partial_1} \bigoplus_{x \in X^{(1)}} K_{n-1}(k(x)) \to \cdots \to \bigoplus_{x \in X^{(n)}} K_0(k(x))$$

$K_0(k(x)) = \mathbb{Z}$

edge map $K_n(X) \to K_n(k(X))$

together get a complex:

$$K_n(X) \to K_n(k(X)) \to \bigoplus_{x \in X^{(1)}} K_{n-1}(k(x)) \to \cdots$$

Gersten conj (true if X is a regular variety)

this is Zariski-locally exact

i.e. true when X is replaced by $\mathcal{O}_{X,x}$

$\Rightarrow$ if we define $\mathcal{K}_n$ to be the presheaf

$$U \mapsto K_n(U) \quad (U \subset X)$$

then we have an exact seq. of sheaves

$$0 \to \mathcal{K}_{n,X} \to \underbrace{\bigoplus_{x \in X^{(0)}} i_{x*} K_n(k(X)) \to \bigoplus_{x \in X^{(1)}} i_{x*} K_{n-1}(k(x)) \to \cdots}$$

$C =$

$C(U)$

complex is exact Zariski locally.

$\Rightarrow$ Hom of computes Zarisk cohom. of $\mathcal{K}_{n,X}$.

$H^p(K_n)$ as cohom of E^1 terms.