Merkurjev-Suslin motivation

A weird occurence of Brauer gp $H^2(F, \mu_\ell)$

Reps of Lie Algebras

$\mathfrak{g}$ semisimple Lie Alg / $\mathbb{C}$

A_n o—o—o— ... —o

B_n o—o— ... o⇒o

C_n o—o— ... o⇐o

A assoc. alg / F f. dim'l

$J(A)$

$A/J(A) \simeq \times M_{n_i}(D_i)$

$\mathbb{C} \quad \simeq \times M_{n_i}(\mathbb{C})$

$\mathfrak{g}$ s.s. Lie alg.

Chose $\mathfrak{h} < \mathfrak{g}$ "Cartan" max'l Abelian

$[x,y]=0$

simple $\mathfrak{h}$ reps are 1 dim'l.

$\mathfrak{h} \circlearrowright \mathbb{C}$

$\mathfrak{h} \to M_1(\mathbb{C})$

$\to \mathbb{C}$

$\mathfrak{h} \circlearrowright \mathfrak{g}$ "adjoint rep"

$\mathfrak{h}^\vee$

gp structure.

$\mathfrak{g} \simeq \bigoplus$ irreps.

$\mathfrak{h} \oplus \underset{\alpha \in \mathfrak{h}^\vee}{\bigoplus} \mathfrak{g}_\alpha$ 1'dim'l

α's called "roots"

Φ

Magic Facts: dim span α's $/\mathbb{C}$ = rank of span$/\mathbb{Z}$

Choose (arbitrary) a "positive" direction
in the $\mathbb{Z}$ span

roots break up into pos & neg.

$\Phi_+ \quad \Phi_-$

Σ_+ "simple roots"
can't be written as nontrivial
sums of others.

give a basis, generate $\Phi/\mathbb{Z}$

natural inner prod on $\mathfrak{g}$ via $\langle x,y\rangle = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))\cdot$ scale factor

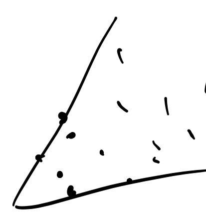
$\mathfrak{g} \circlearrowright \mathfrak{g}$

$\mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{End}(\mathfrak{g})$

Fact of reps of Lie alg is:

Λ = weight space = stuff that $\mathbb{Z}$-pairs roots Φ

Λ_+ = pairs Σ_+ non-neg.
$\longleftrightarrow$ irreps. of $\mathfrak{g}$.



Λ_+

$\mathfrak{g}/F$ F not alg closed?

Λ_+ (over $\bar{F}$) $\rho \in \Lambda_+$

$\updownarrow$?

rep of $\mathfrak{g}$?

$$\Lambda_+ \longrightarrow Br(F)$$

$$\rho \longmapsto A_\rho \quad \text{"Tits Algebra"}$$

$A_\rho \cong M_n(F)$ iff irrep defined over F.

Where are we with the proof?

Goal: H90 K_2 ($\Rightarrow$ MS)

E/F cyclic $E = F(\sqrt[p]{a})$ σ = Gal gen.

$$K_2(E) \xrightarrow{\sigma-1} K_2(E) \xrightarrow{N} K_2(F)$$

Hom here $= V(F)$

Critical thing we needed: $V(F) \hookrightarrow V(F(X))$

$X = SB(A)$ $A = (a,b)_p$

Rough idea: K_2 hard K_1 easier.

want to say: if $\alpha \in K_2(E)$ $N\alpha = 0$

and if $\alpha_{E(X)} = (\sigma - 1)\beta$

want: $\alpha = (\sigma-1)\beta'$

$K_2(X_E)$ edge

$$K_2(E(X)) \xrightarrow{\text{"neighborhood"}} \bigoplus_{\substack{x \in X_E^{(1)} \\ \text{divisor}}} K_1(k(x)) \longrightarrow \bigoplus_{\text{codim } 2} K_0(\,)$$

warlay & chang $\rightsquigarrow$ reduce to understanding

how this term

behaves when moving from

X to X_E.

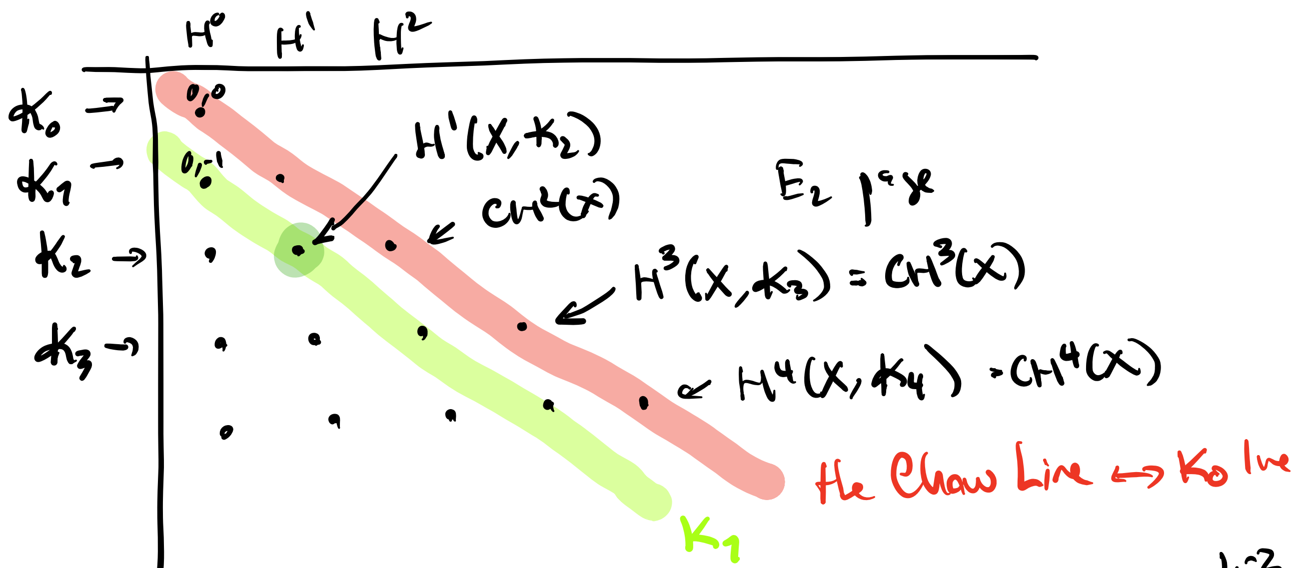
<u>Gersten</u>: the homology here $= H^1(X_E, \mathcal{K}_2)$

reduce to showing $H^1(X, \mathcal{K}_2) \hookrightarrow H^1(X_E, \mathcal{K}_2)$

Today's goal: outline the proof that

$H^1(X, \mathcal{K}_2) \hookrightarrow H^1(X_E, \mathcal{K}_2)$ <u>if</u>

$CH^*(X) \hookrightarrow CH^*(X_E)$.



$$H^1(X,K_2) = E_2^{1,-2} \rightsquigarrow E_3^{1,-2} \rightsquigarrow \cdots \rightsquigarrow E_\infty^{1,-2}$$

squished between CH gps & graded piece of K_1.

relate to K_1

show analogous injectivity for K_1.

graded part of what the K_1 line converges to

First, we'll show that $\mathrm{gr}^i K_1(X) \hookrightarrow \mathrm{gr}^i K_1(X_E)$

w.r.t. to top filtration

<u>Strategy</u>: use fact that

- Know $K_1(X)$ by Quillen (modulo top filtration info)
- Know $K_1(X_E)$ even better ↔ projective space. know top filtration.

Sketch reminder of $K_1(X)$ Quillen

$X = SB(D) \qquad D = (a,b)_p$

$D^{\otimes i} = D \otimes_F D \otimes \cdots \otimes_F D$ i times.

(corresp. to mult. by i in $Br(F)$)

$$K_n(X) = \bigoplus_{i=0}^{p-1} K_n(D^{\otimes i}) \qquad \text{Quillen.}$$

$$\begin{tikzcd}
K_1(X) \arrow[r, dashed, no head] & K_1(D^{\otimes i}) \arrow[dl, "Nrd"'] \arrow[d, equal] \\
F^* & K_1(D) \arrow[l, "Nrd"']
\end{tikzcd}
\qquad
\begin{array}{c} K_1(D^{\otimes 0}) \\ \shortparallel \\ K_1(F) \\ \shortparallel \\ F^* \end{array}$$

Platonov $\Rightarrow$ Nrd is injective.

$$K_1(D^{\otimes i}) \simeq Nrd(D^*)$$
$i > 0$

$$\begin{tikzcd}
K_1(X) \arrow[d, hook, "\text{pullback}"'] \arrow[r, equal] & F^* \arrow[d, hook] \arrow[r, phantom, "\oplus"] & Nrd(D^*) \arrow[d, hook] \arrow[r, phantom, "\oplus"] & Nrd(D^*) \arrow[r, phantom, "\oplus \cdots \oplus"] & Nrd(D^*) \arrow[d, hook] \\
K_1(X_E) \arrow[r, equal] & F^* \arrow[r, phantom, "\oplus"] & F^* \arrow[r, phantom, "\oplus"] & \cdots \arrow[r, phantom, "\oplus"] & F^*
\end{tikzcd}$$

on other hand top filtration on $K_1(X_E) = K_1(\mathbb{P}_E^{p-1})$

$$K_n(\mathbb{P}^{p-1}) = \bigoplus_{i=0}^{p-1} K_n(F)(\gamma-1)^i$$

$$K_n(F) \xrightarrow[\text{pullback}]{} K_n(\mathbb{P}^{p-1})$$

$$K_0(\mathbb{P}^{p-1}) \ni [H] = [\mathcal{O}_H]$$
$$= [\mathcal{O}(-1)] - [\mathcal{O}]$$
$$= \gamma - 1$$

$$F^i K_1(\underset{\mathbb{P}_E^{p-1}}{\overset{\shortparallel}{X_E}}) = (\gamma-1)^i K_1(X_E)$$
$$\shortparallel$$
$$(\gamma-1)^i \cdot E^* + (\gamma-1)^{i+1} \cdot E^* + \cdots$$
$$+ \cdots + (\gamma-1)^{p-1} E^*$$

$$K_1(X) \xrightarrow{f^*} K_1(X_E)$$
preserves top filtration. as does f_*
$$K_1(X_E) \to K_1(X)$$

$$F^i(K_1(X)) \xrightarrow{f^*} F^i(K_1(X_E)) \cap f^* K_1(X)$$
by downward induction on i
$$\shortparallel$$
$$(\gamma-1)^i \operatorname{Nrd}(D^*) + \cdots + (\gamma-1)^{p-1} \operatorname{Nrd}(D^*)$$
$$\simeq \operatorname{Nrd}(D^*)^{p-1-i}$$

gr^i