Vector bundles & their Chern classes

Two goals — Chern classes will be very important later in K-theory

— Help define intersection products

1. understand intersections w/ zero section in vector bundles
2. "deformation to the normal bundle" only work for intersecting $[Z] \cap [W]$ when either Z or W is "regularly embedded"

3. in general $[Z] \cap [W]$ is done via intersecting $Z \times W$ w/ $\Delta \subset X \times_k X$

   regarding this as in $CH(\Delta) = CH(X)$

   works if $\Delta \hookrightarrow X \times_k X$ is regularly embedding

   true if $X/_k$ is <u>smooth</u>.

<u>Today:</u> $\cap$ w/ 0 section & chern classes

2 step process:
1. line bundles
2. extending to general v.b.'s.

Step 1: The chern class of a line bundle

Cartier divisor classes $\longleftrightarrow$ Iso classes of line bundles

(work with X = variety over a field)

(Hartshorne II.6 or Fulton App. B)

Given a line bundle L on X (= locally free sheaf of rk 1)

can choose open cover U_i s.t. $L|_{U_i} \xrightarrow[\varphi_i]{\sim} \mathcal{O}_{U_i}$

and isomorphisms on overlaps

$$\mathcal{O}_{U_{ij}} \xleftarrow[\varphi_i|_{U_{ij}}]{\sim} L|_{U_i}|_{U_{ij}} = L|_{U_{ij}} = L|_{U_j}|_{U_{ij}} \xrightarrow[\varphi_j|_{U_{ij}}]{\sim} \mathcal{O}_{U_{ij}}$$

$$g_{ij} \in \mathcal{O}_X(U_{ij})^*$$

$i=0$ some index

$$f_i = g_{i,0} \qquad g_{ij} = g_{ji}^{-1} \qquad g_{ij}g_{jk} = g_{ik}$$

$$f_i/f_j = g_{i0}/g_{j0} = g_{i0}g_{j0}^{-1} = g_{i0}g_{0j} = g_{ij}$$

$$\mathcal{O}_X(U_{ij}) \hookrightarrow k(X)^*$$

Define a Cartier divisor via: $D = (U_i, f_i)_i$

Conversely, given a Carter divisor
$$D = (U_i, f_i)$$
define $\mathcal{O}_X(D) = \{$rat'l fns w/ poles no worse than $D\}$

$$\mathcal{O}_X(D)(U) = \{ f \in k(X) \mid \operatorname{div}(f)|_U + D|_U \geq 0 \}$$

alternately, subsheaf of $K(X)^*$ which on U_i consists of $f_i^{-1}\mathcal{O}_{U_i}$
(affine open cover)

Remarks: $\mathcal{O}_X(D)^* = \mathcal{O}_X(-D)$
if D is effective (i.e. f_i are regular fcns)
then $\mathcal{O}_X(-D) \cong \mathcal{I}_D$ = shf of fcns vanishing along D

Define "intersecting"
if D is a Cartier divisor, we define $c_1(D)$
to be the operator
$$CH_i(X) \longrightarrow CH_{i-1}(X)$$
$$[V] \longmapsto c_1(D) \cap [V]$$
$$= [\text{Cartier divisor}$$

assoc. to $\mathcal{O}_X(D)|_V$

i.e. if we've given a line bundle

L/X, can choose D s.t. $L \cong \mathcal{O}_X(D)$

$D = (U_i, f_i)$ $\quad [D] = $ "$\mathrm{div}(f_i)$"

= Weil divisor associated to D

"$c_1(L)$" $\longleftrightarrow [D] \in CH_{\dim X - 1}(X)$

for any $V \subset X$, can consider $[L|_V] \in CH_{\dim V - 1}(V)$

$$c_1(L) \cap [V] = (i_{V \hookrightarrow X})_* [L|_V]$$

we write interchangeably $c_1(L)$ or $c_1(D)$

$$c_1(L) \cap \alpha = \sum n_i \, c_1(L) \cap [V_i]$$

extend linearly

if $\alpha = \sum n_i [V_i]$

write this as $\quad c_1(L) \cap \alpha = c_1(D) \cap \alpha = [D] \cdot \alpha$

Another property: given $V \overset{i}{\subset} X$ subvariety, $D \subset X$ Cartier

can consider $[\text{Weil divisor assoc. to } i^* \mathcal{O}_X(D)]$

$c_1(i^* \mathcal{O}_X(D)) \cap [V] \in CH_{\dim V - 1}(V)$

proper pushforward.

$$i_* \left((c_1(i^* \mathcal{O}_X(D)) \cap [V] \right) = D \cdot [V]$$

There are a bunch of things to say / check about this.

Some properties

- $[D] \cdot (\alpha + \beta) = [D]\alpha + D[\beta]$
- $([D] + [D']) \cdot \alpha = [D] \cdot \alpha + [D'] \cdot \alpha$

"$CH(X)$ is a module over the $c_1[D]$'s"

- projection formula: if $f: X' \to X$ proper

$f^*: \operatorname{Pic} X \to \operatorname{Pic} X'$ which allows $c_1(L)$'s for L/X to act both on $CH(X)$ & $CH(X')$.

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*\alpha$$

- if $X' \to X$ flat then $f^*D \cdot f^*\alpha = f^*(D \cdot \alpha)$

- $D \cdot [D'] = D' \cdot [D]$
- $D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$

Self intersection cultural comment

D Cartier divisor in X, write D to also mean
effective
corresponding Weil divisor

$$D \xhookrightarrow{i} X$$

How to intersect D with itself?

$$c_1(i^* \mathcal{O}_X(D)) \cap D$$

$$i^* \mathcal{O}_X(D) = i^*(\mathcal{I}_D^\vee) = (i^* \mathcal{I}_D)^\vee$$

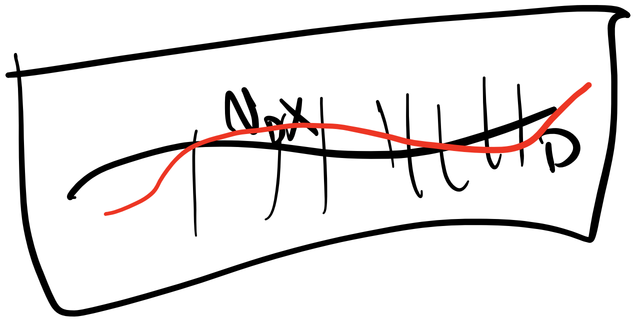
$$i^* \mathcal{I}_D = \mathcal{I}_D \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I}_D = \mathcal{I}_D / \mathcal{I}_D^2$$ as an $\mathcal{O}_X/\mathcal{I}_D$ mod

$$M \otimes_R R/I = M/IM$$

Hartshorne calls $\mathcal{I}_D/\mathcal{I}_D^2$ the conormal bundle

and its dual $(\mathcal{I}_D/\mathcal{I}_D^2)^\vee = N_D X$
normal bundle.

$$c_1(D) \cap [D] = c_1(N_D X) \cap [D] \text{ in } CH(D)$$



$N_D X$
$\downarrow$ inv. shf
D (if D Cartier)

$$c_1(N_D X) \cap [D] \equiv \text{"} D \cdot D \text{"}$$

General fact: if $Z \subset X$ loc. cut out by regular sequence
then $\mathcal{I}_Z/\mathcal{I}_Z^2$ a loc. free shf.

Reality check fact: (meaning of c_1) $\quad c_1(L) \equiv c_1(L) \cap [X]$
if L is a line bundle on a variety X
an $s \in \Gamma(X, L) \setminus \{0\}$
then can define $\operatorname{div}(s)$
and we have $[\operatorname{div}(s)] = c_1(L) \equiv c_1(L) \cap [X]$

Very important property:
if $V \xrightarrow{\pi} X$ is a vector bundle
i.e. $V = \underline{\operatorname{Spec}}_{\mathcal{O}_X}(\operatorname{Sym}^\bullet(\mathcal{F}))$
loc. free rk n sheaf

then $CH_i(X) \xrightarrow[\pi^*]{} CH_{i+r}(V)$

is an isomorphism.

" $(\pi^*)^{-1}$ = intersection w/ 0-section "