Last time, defined

$c_1(D) \cap \alpha = c_1(L) \cap \alpha = D \cdot \alpha$

D a Cartier divisor whose class corresponds to the line bundle L

Two things today:

- $\boxed{\begin{array}{l}\text{Isomorphism for vector bundles: } \\ V = \text{v.b. of rk } r \text{ over } X \\ \text{then } CH_i(X) \xrightarrow[\sim]{\pi^*} CH_{i+r}(V) \\ \text{is an isomorphism.}\end{array}}$ $\quad \begin{array}{c} V \\ \downarrow \pi \\ X \end{array}$
- Definition of higher chern classes
- Deformation to normal cone

Aside: we defined pullback $f: X \to Y$ flat

$f^*: Z(Y) \to Z(X)$

$[Z] \longmapsto [f^{-1}(Z)]$ ← scheme-theoretic inverse image

in practice: most important one

$U \hookrightarrow X$ open immersions are flat

$$\begin{array}{ccc} E & \longrightarrow & X \\ \uparrow & & \uparrow \\ E\times_X U_i & \longrightarrow & U_i \\ \| & & \\ U_i\times F & & \end{array}$$ locally a product (Zariski)

Similarly $f: X \to Y$ proper $f_*: Z_i(X) \to Z_i(Y)$

most important examples:

- closed imbeddings $Z \hookrightarrow X$
- projective morphisms

$$\begin{array}{ccc} X & \longrightarrow & pt \\ \uparrow & & \\ \text{proj var.} & & \end{array}$$

Theorem (Prop 1.9, Thm 3.3(a) in Fulton)

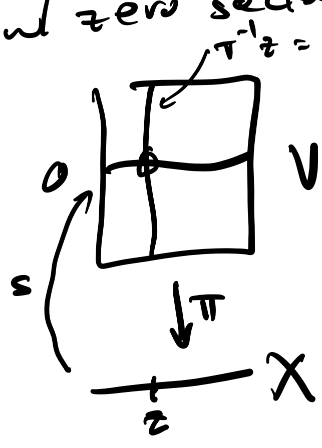
if $X/_k$ variety, V a rk r v.bundle over X then

$$CH_i(X) \xrightarrow[\pi^*]{\sim} CH_{i+r}(V)$$

<ins>Consequence</ins>: can define "intersection w/ zero section of V"

$$CH_{i+r}(V) \xrightarrow[(\pi^*)^{-1}]{} CH_i(X)$$

s^*



$(\pi^* z) \cap s = z$ $\qquad$ $(\underbrace{V|_z}_{\pi^{-1}z = \pi^* z}) \cap s = z$

Proof (sketch): π^* is surjective

In case: $X = \operatorname{Spec} k$ is a pt.

$$CH_0(pt) = CH_1(\mathbb{A}^1)$$

$$Z_1(\mathbb{A}^1) = \mathbb{Z}\cdot[\mathbb{A}^1] = \mathbb{Z}$$

✓

in general, reduce to case X affine, V/X is $\mathbb{A}^1_X$

induct on $\dim X$.

on full statement

Case of dim d choose $U \subset X$ affine open s.t. $V|_U \cong \mathbb{A}^n_U$

prop. pushforward, flat pullback

$$CH_i(X \setminus U) \to CH_i(X) \to CH_i(U) \to 0$$

$$Z_i(X) \to Z_i(U)$$

$$\alpha \mapsto 0$$

W_j dim $i+1$

$f_j \in k(W_j)^*$

$\sum \operatorname{div}_U(f_j) = \alpha|_U$

consider $\overline{W_j} \in X$

$k(\overline{W_j}) = k(W_j)$

$\sum \operatorname{div}_X(f_j) = \beta$

$\beta - \alpha \in Z_i(X \setminus U)$

smaller dim

$$\begin{tikzcd}
{} \arrow[d] & & & \\
CH_i(X \setminus U) \arrow[r] \arrow[d, "\wr"] & CH_i(X) \arrow[r] \arrow[d] & CH_i(U) \arrow[r] \arrow[d, "\pi^* \wr"] & 0 \\
CH_i(V|_{X\setminus U}) \arrow[r] & CH_i(V) \arrow[r] & CH_i(V|_U) \arrow[r] & 0
\end{tikzcd}$$

$$V|_U = \mathbb{A}^n_U$$

so by induction

if true for triv bundles

then by diag chase, middle is also an iso.

reduced to case of $\mathbb{A}^r_X \to X = \operatorname{Spec} A$

notice that $CH_i(X) \to CH_{i+r}(\mathbb{A}^n_X)$ factors as

$$CH_i(X) \to CH_{i+1}(X \times \mathbb{A}^1) \to CH_{i+2}(X \times \mathbb{A}^1 \times \mathbb{A}^1) \to \cdots$$

So reduce to case of $\mathbb{A}^1 \times \operatorname{Spec} A \to \operatorname{Spec} A$

$$A[t] \hookleftarrow A$$

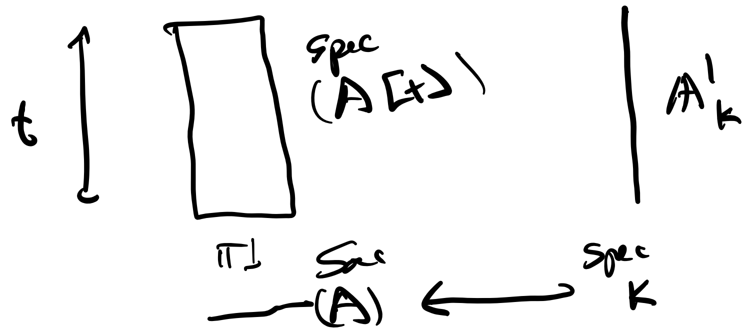
given a prime $p \subset A$ then $\pi^{-1}(p) = pA[t]$

so $\pi^*[p] = [pA[t]]$ $\quad A/p$ domain $\Rightarrow A/p[t]$

$A/p \otimes_A A[t]$ is a domain

to show π^* is surjective (in CH), want to start w/

$Q \triangleleft A[t]$ induct on dim A



consider $\pi(Q) = Q \cap A$

☆ if cycle $[Z] \in Z_{i+r}(A^1_X)$ such that

$\overline{\pi(Z)} \subsetneq X$ then consider

$$\begin{tikzcd}
{[Z]} \arrow[rrr, bend left, mapsto] & A^1_{\overline{\pi(Z)}} \arrow[r, hook, "j"] \arrow[d, "\pi"] & A^1_X \arrow[d, "\pi"] & {[Z]} \\
& \overline{\pi(Z)} \arrow[r, hook, "i"'] & X &
\end{tikzcd}$$

by induction on dim, $[Z] = \pi^* \beta$

via some Lemmas in Fulton

$$[Z] = j_* \pi^* \beta = \pi^* i_* \beta = \pi^*(i_* \beta)$$ ✓

WLOG, $Z \to X$ is dominant. $K = \operatorname{frac} A$

$\Longrightarrow$ $Q \cap A = (0)$ so $QK[t] \neq K[t]$ is a nontrivial proper ideal.

$Q \triangleleft A[t]$ prime s.t. $Q \cap A = (0)$ (why??)
nontriv.

$S = A \setminus \{0\}$

$Q \cap (A \setminus \{0\})$

$QK[t] = r(t)K[t]$

$r(t) \in K[t] \subset K(t) = k(\mathbb{A}^1_A)$

consider divisor of $r(t)$ as an element of $k(\mathbb{A}^1_A)^*$

in $\mathbb{A}^1_A$ there are 2 kinds of primes

$$\begin{array}{c} \mathbb{A}^1_A \\ \downarrow \\ \operatorname{Spec} A \end{array}$$

1. primes which dominate Spec A (map to gen. pt downstairs)
2. primes that don't.

primes in $K[t]$

by construction, $v_{(\text{type 1 prime})}(r(t))$ only $\neq 0$ at Q.

$\operatorname{div}_{\mathbb{A}^1_A}(r(t)) = [Q] +$ some other primes which don't dominate A.

in $CH_{i+1}(\mathbb{A}^1_A)$ $\quad [Q] = [Q] - \operatorname{div}(r(t))$

$= \sum n_i [\text{primes don't dom}]$

but now, as in ☆, done by induction!

<u>Goal</u>: define pullback (Gysin) map associated to a regularly embedded closed subvariety

$$Z \xhookrightarrow{i} X \quad \text{codim } c$$

$$i^*: CH_n X \longrightarrow CH_{n-c} Z$$

idea: want to convert an intersection problem $i^* W$ to a "linearized version"

$W \subset X$

$$Z \xhookrightarrow[s]{\text{zero section}} N_Z X$$

$\cup$ Cone associated to W, $W \cap Z$

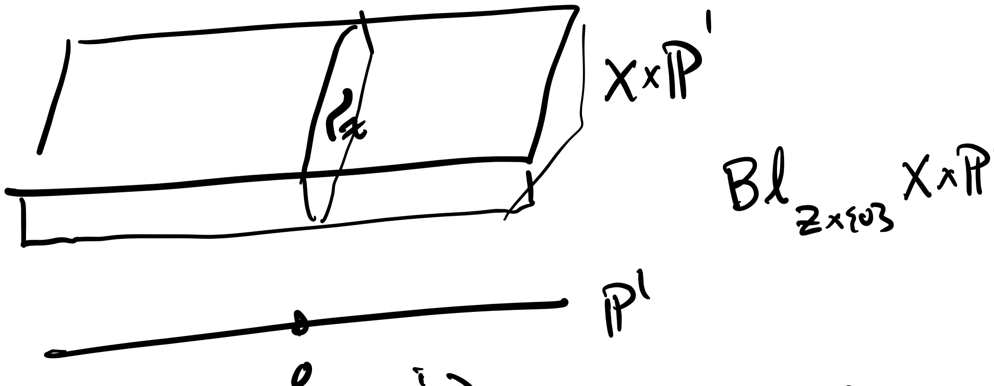
$\tilde{W}$

$$i^*[W] = s^*([\tilde{W}])$$

Outline of process:

construct a family parametrized by $\mathbb{P}^1$

most fibers will be X & special will be $N_Z X$



$$\begin{tikzcd}
X \setminus Z \arrow[r, hook] \arrow[d, "\wr"] & Bl_Z X \arrow[d] & \mathrm{Proj}_{\mathcal{O}_Z}\left(\bigoplus \mathcal{I}_Z^i/\mathcal{I}_Z^{i+1}\right) \arrow[l, hook'] \arrow[d] \\
X \setminus Z \arrow[r] & X & Z \arrow[l]
\end{tikzcd}$$

$Bl_Z X = \mathrm{Proj}\left(\bigoplus \mathcal{I}_Z^i\right)$

if $Z \hookrightarrow X$ locally cut out by reg sequence

then $\bigoplus \mathcal{I}_Z^i/\mathcal{I}_Z^{i+1} \underset{\mathcal{O}_Z\text{-alg}}{\simeq} \mathrm{Sym}^i_{\mathcal{O}_Z}\big(\underbrace{\mathcal{I}_Z/\mathcal{I}_Z^2}_{\substack{\text{loc. free}\\ \text{of rk} =\\ \text{codim } Z}}\big)$

$$\left(\mathcal{I}_Z/\mathcal{I}_Z^2\right)^\vee = N_Z X$$

$$\mathrm{Proj}\left(\bigoplus \mathcal{I}_Z^i/\mathcal{I}_Z^{i+1}\right) = \mathrm{Proj}\left(N_Z X\right)$$

V/k vector space $\quad \mathbb{A}(V)$ scheme whose k-pts are naturally $= V$

$$\mathbb{A}(V) = \mathrm{Spec}(S^\bullet V^*)$$