Intersection products via deformation to normal cone

"Specialization to normal cone"

$Z \hookrightarrow X$ regularly embedded

$V \hookrightarrow X$ closed subscheme, get a closed embedding

$$C_{V\cap Z} V \hookrightarrow C_Z X = N_Z X \quad \text{normal bundle.}$$

define specialization map $\qquad c = \text{codim of } Z \text{ in } X$

$$\begin{array}{ccccccc} Z_k(X) & \xrightarrow{sp} & Z_k(N_Z X) & \xrightarrow{\cap \text{ zero section}} & CH_{k-c}(Z) & \to & CH_{k-c}(X) \\ [V] & \longrightarrow & [C_{V\cap Z} V] & \longrightarrow & [V]\cap[Z] & \to & [V]\cdot[Z] \end{array}$$

In fact: sp well defined map $CH_k(X) \to CH_k(N_Z X)$

this comes from "deformation to normal cone"

Construct a family

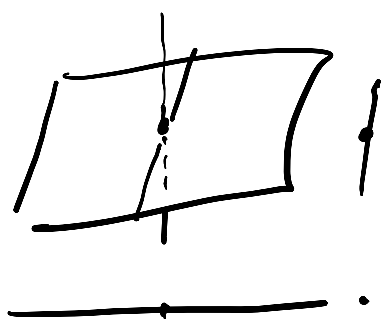
$$\begin{tikzcd} V \arrow[r, dashed] & \bar{V} \arrow[r] & \cdots \\ X \arrow[r, hook] \arrow[d] & M_Z X \arrow[d, "\text{flat}"] & N_Z X \arrow[l, hook'] \arrow[d] \\ * \arrow[r, hook, "\text{typical}"'] & \mathbb{A}^1 & 0 \arrow[l, hook'] \end{tikzcd}$$

Definition of $M_Z X$

$$\begin{array}{c} Bl_{Z\times\{0\}}(X\times\mathbb{A}^1) \\ \downarrow \\ X\times\mathbb{A}^1 \hookleftarrow Z\times\{0\} \end{array}$$

X

$\mathbb{A}^1\times\mathbb{P}^1$



$Bl_{\{0\}\times\{0\}}\,\mathbb{A}^1\times\mathbb{P}^1$

$\mathbb{A}^1$

0

central fiber two components.

always have inv. image of $Z\times\{0\}$ is a divisor in $Bl_{Z\times\{0\}} X\times\mathbb{A}^1$

other component of inv. im. of 0 is the proper transform of inv. im ~~cof~~ 0

exceptional divisor:

$$\mathrm{Proj}\left(\bigoplus \mathcal{I}^n_{Z\times\{0\}}/\mathcal{I}^{n+1}_{Z\times\{0\}}\right)$$
$$\downarrow$$
$$Z\times\{0\}$$

$\overset{\shortparallel}{\mathbb{P}}(N_Z X\oplus 1)$

$Z \hookrightarrow V \hookrightarrow X$

$Bl_Z X \hookleftarrow$ prop. trans. of V

$\downarrow \pi$

X

$= \overline{\pi^{-1}(V\setminus Z)}$

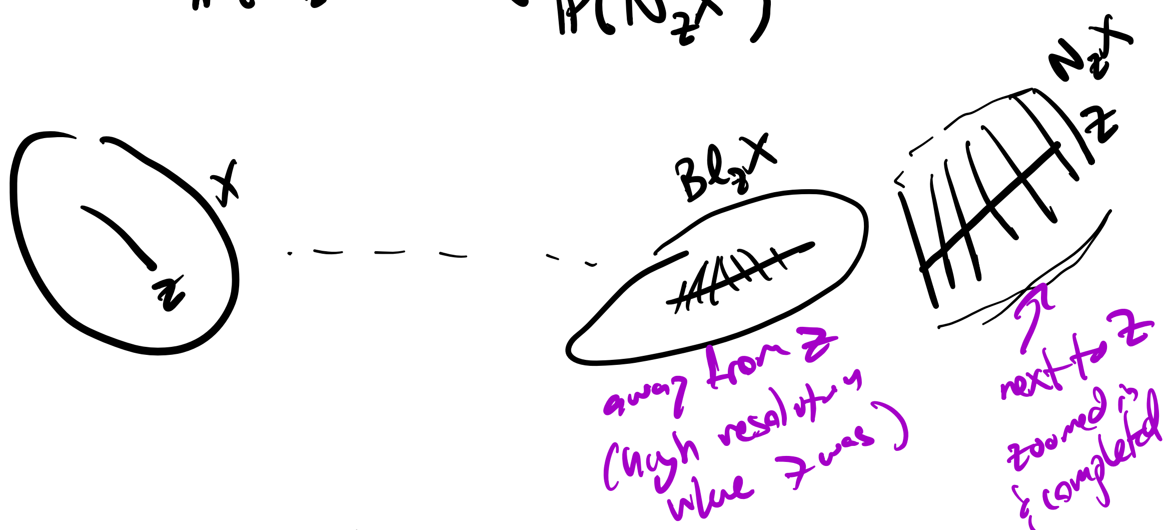
$= Bl_Z V$ (Fulton App B possibly w/ hypothesis) in our case, this is $Bl_{Z\times\{0\}}(X\times\{0\}) \cong Bl_Z X$

$$\begin{tikzcd} Bl_Z X \arrow[d] & \mathbb{P}(N_Z X) \arrow[l, hook'] \arrow[d] \\ X & Z \arrow[l, hook'] \end{tikzcd}$$

Pieces of exceptional divisor of $Bl_{Z\times\{0\}} X\times\mathbb{A}^1$

comps:

$$\begin{tikzcd} \mathbb{P}(N_Z X \oplus 1) & & Bl_Z X \\ \mathbb{P}(N_Z X) \arrow[u] \arrow[r, equal] & \mathbb{P}(N_Z X) \arrow[ur, "\text{excep}"'] & \end{tikzcd}$$



$$M_Z X = \left(Bl_{Z\times\{0\}} X\times\mathbb{A}^1\right) \setminus Bl_Z X$$

at the central fiber $= \mathbb{P}(N_Z X\oplus 1)\setminus \mathbb{P}(N_Z X)$

$= N_Z X$

$$
\begin{tikzcd}
\mathbb{P}(N_Z X \oplus 1) \sqcup_{\cancel{\mathbb{P}(N_Z X)}} \cancel{Bl_Z X} \arrow[r, hook] \arrow[d] & Bl_{Z \times \{0\}} X \times \mathbb{A}^1 \arrow[d] & X \arrow[l, hook'] \arrow[d] \\
0 \arrow[r, hook] & \mathbb{A}^1 & * \arrow[l, hook']
\end{tikzcd}
$$

$M_Z X$

Intersection ring (X smooth variety over a field k)

$CH^*(X)$ ring structure on cycle classes is defined by:

$[V] \cdot [W]$ $\quad V, W \hookrightarrow X$ closed subvarieties

$\Delta : X \hookrightarrow X \times_k X$ is a regular embedding.

(more generally: if $f : X \to Y$, Y smooth then

$\gamma_f : X \to X \times Y$ is a regular embedding)

define $[V] \cdot [W] = \Delta^*([V \times W])$

$\qquad = \{x \mid (x,x) \in V \times W\}$

Also gives general pullbacks (not necessarily flat!)

if $f: X \to Y$ morphism, X, Y smooth.

$$f^*(\alpha) = \gamma_f^*([X] \times \alpha)$$

$\alpha \in CH^{\ell} Y$

Chow theory for today.

Chern classes

Recall: defined $c_1(D) = c_1(L) \in CH^1(X)$

$$L = \mathcal{O}(D)$$

via: $c_1(L) \equiv c_1(L) \cap [X] = [D]$

great way of expressing info about a line bundle

$$L \rightsquigarrow c_1(L) = [D] \in CH^1(X) = \operatorname{Pic} X$$

X regular

Higher dim?

E vbundle what information can $CH(X)$ see about E?

Special case $E = \bigoplus_{i=1}^{r} L_i$

Notice that any symmetric poly expression in $c_1(L_i)$'s turns out to give a well defined op on rk r bundles.

$$c_1(E) = c_1(L_1) + \cdots + c_1(L_r) \in CH^1$$

$$c_2(E) = \sum_{i<j} c_1(L_i)\cdot c_1(L_j) \in CH^2$$

$$c_\ell(E) = \sum_{i_1<\cdots<i_\ell} c_1(L_{i_1})\cdots c_1(L_{i_\ell}) \in CH^\ell$$

"The Chern classes"

$$\begin{tikzcd}
\text{Vectr bundles} \arrow[rr, squiggly, "\text{chern classes}"] \arrow[d, "\text{``chern class''}"'] & & \text{Chow gps} \\
\text{othr thy (Ab. gps, rgs!)} \arrow[urr] & &
\end{tikzcd}$$

some same props, but losing less info.

$$\begin{tikzcd}
\text{Schemes}/k \arrow[rr, squiggly, "H^*"] \arrow[d, squiggly] \arrow[dr, squiggly] & & \text{Cohom. thys} \\
\text{Motives} \arrow[r] & \text{Motivic Coh} \arrow[ur] &
\end{tikzcd}$$

Def of higher Chern classes (splitting principle)

(all varieties smooth)

<u>Goal</u>: define a notion of $c_\ell(E)$ $\ell \leq \operatorname{rk} E$

s.t. • if $E \cong L_1 \oplus \cdots \oplus L_r$ then

$c_\ell(E) =$ elem. symm. poly of deg ℓ in $c_1(L_i)$'s

• if $X \xrightarrow{f} Y$ E/Y then

$f^* c_\ell(E) = c_\ell(f^* E)$

more generally if we have

$E \supset E^1 \supset \cdots \supset E^r = 0$ codim E^i in E is i

$L_i = E^i / E^{i+1}$, same def for $c_\ell(E)$ as $c_\ell(\oplus L_i)$

Construction for general bundles is then as follows:

<u>Splitting principle Lemma</u>: Given E/X v.b.

then $\exists\ \pi : \tilde{X} \longrightarrow X$ s.t.

- $\pi^* : CH\,X \longrightarrow CH\,\tilde{X}$ injective
- $\pi^* E \simeq \cancel{\bigoplus_{i=1}^{n} L_i}$ has a filtration
  $\pi^* E = \tilde{E}^0 \supset \tilde{E}^1 \supset \cdots \supset \tilde{E}^n = 0$

if you believe this, then have to have

$$\pi^* c_\ell(E) = c_\ell(\pi^* E) = c_\ell(\oplus L_i)$$

= determined by axioms

$\pi^* c_\ell(E)$ uniquely determines $c_\ell(E)$.

<u>Pf of sp. lemma (sketch)</u>

induct on rank E (rk 1 ✓)

in general, given E/X

$$\begin{array}{c} \mathbb{P}(E) \\ \downarrow \\ X \end{array} \qquad \mathcal{O}(-1)_{\mathbb{P}(E)} \hookrightarrow E_{\mathbb{P}(E)} \to T_{\mathbb{P}(E)/X}$$

(Euler sequence)

$$E_{\mathbb{P}(E)} = \{(v,\ell) \mid v \in E,\ \ell \subset E \text{ line}\} \hookleftarrow \text{subbundle } \mathcal{O}(-1)_{\mathbb{P}(E)} = \{(v,\ell) \mid v \in \ell\} \quad \ell \subset E \text{ lines}$$

$$\begin{array}{ccc} \mathcal{O}(-1)_{\mathbb{P}(E)} & & \{(v,\ell) \mid v\in\ell\} \\ \downarrow & & \downarrow \\ \mathbb{P}(E) & & \ell \end{array}$$