Today: Panin-Smirnov - Groth. Hirzebruch - Riemann-Roch Thm.

"Oriented Cohomology theory"

Use the def from "Algebraic Cobordism" (Levine-Morel)

Today: Consider smooth varieties over fields. Sm_k

Def A ring cohomology pretheory is a contravariant functor $Sm_k \xrightarrow{A} GRings$ ($\mathbb{Z}$-graded rings)

s.t.
- $A(X \sqcup Y) = A(X) \times A(Y)$
- If $E \xrightarrow{\pi} X$ is a vector bundle then $A(X) \to A(E)$ is an isomorphism
- $A(\emptyset) = (0)$

[Side comment: $A = CH$ we get pullbacks via:

$X \xrightarrow{f} Y \qquad X \xrightarrow{id \times f} X \times Y \xrightarrow{\pi} Y$

flat pullback

pullback via normal cone

Claim: if X, Y smooth $\Rightarrow$ $id \times f$ is a reg. imbedding.]

"unoriented" cohom. thy

What's an orientation?

two equivalent formulations:

either

- pushforwards for projective morphisms
- first chern classes for line bundles (⇝ chern classes for vb's...)

Def Oriented coh. thy (on Sm/k) is the data of

1) A mg coh. pretheory A
2) For each proj. morphism $f: Y \to X$ of rel dim d, a hom of $A(X)$-modules

$$f_*: A^n(Y) \longrightarrow A^{n+d}(X)$$

note: $A^*(X) = A(X)$

$A(Y)$ is an $A(X)$ module via $f^*: A(X) \to A(Y)$

(i.e. if $\alpha \in A(Y)$, $\beta \in A(X)$

$$f_*(\beta \cdot \alpha) = \beta \cdot f_*(\alpha)$$

$$\parallel$$

$$f_*(f^*\beta \cdot \alpha)$$

) "projection formula"

[**Def**: $f: Y \to X$ has rel. dim d if for each $y \in Y$, (dim of Y at y) = (dim of X at x) + d)

$\mathbb{P}^d \times X \to X$ rel dim d

$D \hookrightarrow X$ div in X rel dim -1]

$$\left[\begin{tikzcd} \mathbb{P}^d \times D \arrow[rr] \arrow[dr] & & X \\ & D \arrow[ur] & \end{tikzcd} \quad \text{rel dim } d-1 \right]$$

Such that: • $id_* = id_{A(X)}$

• if $X, Y \to Z$ are transverse morphisms

and we consider

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

w/ f projective of rel dim d.

then $g^* f_* = (f')_* (g')^*$

(transverse means:

$\mathrm{Tor}_q^{\mathcal{O}_Z}(f_*\mathcal{O}_X, g_*\mathcal{O}_Y) = 0 \quad q > 0$)

• Let E be a rk n v.bundle on X,

$\mathcal{O}(1) / \mathbb{P}(E)$ has a canonical section $s: \mathbb{P}(E) \to \underline{\mathcal{O}(1)}$

seque / line bundle ass. to $\mathcal{O}(1)$

define $\xi_E = s^{*} s_{*} 1_{A(\mathbb{P}(E))}$

"$c_1 \mathcal{O}(-1)$"

then $A(\mathbb{P}(E)) \simeq \bigoplus_{i=1}^{n-1} A(X) \cdot \xi^i$

as $A(X)$ modules (free)

Important fact:

equivalently, can define this as a pre-theory with a notion of $c_1(\mathcal{L})$ $\mathcal{L}$ line bundle s.t. various axioms.

How to get pushforwards:

$c_1(\mathcal{L}) \longleftrightarrow \cap$ w/ divisors

morally reasonable that since reg. imbedded looks like locally cut out by multip divisors, that can iteratively use $c_1(\mathcal{L})$'s.

real answer: Fulton. MacPherson trick
of degeneration to normal cone.

Statement of Groth. R-R concerns comparison of pushforwards in two theories:

Suppose we have a morphism of (oriented) coh. thys

$$A^* \xrightarrow{\varphi} B^*$$

which doesn't necessarily preserve pushforwards.

Goal: describe the relation between pushfwds.

i.e. correct the failure of commutativity of

$$X \xrightarrow{f} Y$$

$$\begin{tikzcd} A(X) \arrow[r, "\varphi_X"] \arrow[d, "f_* = f_A"'] \arrow[dr, phantom, "\times"] & B(X) \arrow[d, "f_* = f_B"] \\ A(Y) \arrow[r, "\varphi_Y"'] & B(Y) \end{tikzcd}$$

Illustrate to consider the case where $A \cong B$ as rings but w/ diff pushfwds.

Then: Q: how to describe all possible orientations?

Turns out that for the purposes of oriented coh. theories,

$\exists$ a "universal line bundle" $\qquad c_1(\mathcal{O}(-1))$

$$\mathbb{P}^1_k \hookrightarrow \mathbb{P}^2_k \hookrightarrow \mathbb{P}^3_k \hookrightarrow \cdots \hookrightarrow \mathbb{P}^\infty_k$$

$$\underset{\mathbb{P}^2_k}{\mathcal{O}(-1)} \; - \; \underset{\mathbb{P}^3}{\mathcal{O}(-1)}$$

line bundle here

= comp. collection of line bundles on all spaces

$$\operatorname{Pic} \mathbb{P}^\infty \cong \varprojlim_n \operatorname{Pic} \mathbb{P}^n$$

$$VB(\mathbb{P}^\infty) \cong \varprojlim VB(\mathbb{P}^\infty)$$

$$A(\mathbb{P}^\infty) = \varprojlim A(\mathbb{P}^n)$$

$$\operatorname{Pic} \mathbb{P}^\infty \dashrightarrow \operatorname{Pic} \mathbb{P}^{n+1} \to \operatorname{Pic} \mathbb{P}^n \to$$

$$\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$$

$$A(\mathbb{P}^\infty) \dashrightarrow A(\mathbb{P}^{n+1}) \to A(\mathbb{P}^n) \to \cdots$$

$$A(\mathbb{P}^n) = \bigoplus_{i=0}^{n-1} A(pt)\cdot \xi^i = A(p)[\xi]/(\xi^n - \cdots)$$

uses orientations!

same limit as $\lim_{\leftarrow n} A(pt)[\xi]/(\xi^n)$

$\parallel$

$\xi \in A(pt)[[\xi]]$

$c_1^A(\mathcal{O}(-1)) = \xi$

Any orientation on $A^{\text{unoriented}}$ ← the underlying spectrum for A
is determined by a choice of a power series generator
$\xi \in A(\mathbb{P}^\infty)$ which we can use to
identify $A(\mathbb{P}^\infty)$ w/
$A(pt)[[\xi]]$

turns out that any choice of a top gen for $A(pt)[[\xi]]$
corresponds to a uniquely defined other orientation!

$\xi + \xi^2 + 4\xi^5$

$A \leftarrow A'$ ← thing w/ = ring diff orientation

$A(\mathbb{P}^\infty) \leftarrow A'(\mathbb{P}^\infty)$

$\xi_A \leftarrow \xi_{A'}$

$\varphi(\xi_A) = \xi_A + \xi_A^2 + \cdots$

$\varphi(\xi_A')/\xi_A \in A(pt)[[\xi_A]] \cong A(pt)[[t]]$

td(t)