Last time:

Introduced oriented cohom. theories

Cohom. pretheory

$$A : (Sm/_k)^{op} \longrightarrow GRngs$$

extra structure : orientation

various definitions

pushforwards for proj. morphisms

chern classes $c_1(L)$

ω = orientation on A (A, ω) "oriented thy"

Recall last time: given (A, ω), gives an identification of $A(\mathbb{P}^\infty_k) \simeq k[[\xi]] \simeq k[[t]]$ $\xi \leftrightarrow c_1^\omega(\mathcal{O}(-1))$

given another orientation, ω', then gives us to an iso.

$$\begin{tikzcd} & A(\mathbb{P}^\infty) \arrow[dl, "\omega'", "\sim"'] \arrow[dr, "\omega", "\wr"'] & \\ A(pt)[[t]] \arrow[rr, bend right] & & A(pt)[[t]] \\ & & a(t) \end{tikzcd}$$

say $a(t) = a_1 t + a_2 t^2 \cdots$
$a_i \in A(pt)^{\times}$

$a(t)/t = a_1 + a_2 t$

$\left(\overset{\shortparallel}{r(t)} \right)$

Part 1: calculus of "conversion factors"

Suppose $r(t) \in A(pt)[\![t]\!]$

(A, ω) oriented coh. thy

Prop: $\forall\, r(t) \in A(pt)[\![t]\!]$ $\exists$ a unique assignment

tply v. bundles $E/X \rightsquigarrow r(E) \in A(X)$ s.t.

1) $r(L) = r(c_1^\omega(L))$

2) given $f : X \to Y$ then $f^*(r(E)) \overset{\shortparallel}{} r(f^*E)$

3) $r(E) = r(E_1) r(E_2)$ where

$0 \to E_1 \to E \to E_2 \to 0$ ses.

4) if $r(t) \in A(pt)[\![t]\!]^\times$ then

$r(E) r^{-1}(E) = 1$ in $A(X)$

$c_t = 1 + c_1 t + c_2 t^2 + \dots$

One useful operation pushforward for closed embeddings.

"Gysin maps"

$Y \xhookrightarrow{i} X$

$r(t)$ ω

$i_*^{\omega,r} = i_*^{\omega} \circ r(N) : A(Y) \longrightarrow A(X)$

$N = N_{X/Y}$ (imagine $r(t) = 1 + HOT$)

this gives good system of gysin maps.

In more generality: if $f: Y \longrightarrow X$ is a proj morphism an $r(t)$ invertible, then

$$f_*^{\omega,r} : A(Y) \longrightarrow A(X)$$
$$\alpha \longmapsto r(T_X) \cdot f_*^{\omega}(\alpha \cdot r^{-1}(T_Y))$$

gives a new orientation.

Moreover: if ω' is any other orientation,

then $\exists\, r(t)$ s.t. $f_*^{\omega,r} = f_*^{\omega'}$
invertible.

<u>Final step</u>: how to get r ?

given a morphism of pretheories $\varphi : A \to B$
where A, B are oriented.

then let $r(\xi_B) = \xi_B / \varphi(\xi_A) \in B(\mathbb{P}^\infty) = B(pt)[[\xi_B]]$

$$= td_\varphi(\xi_B)$$

$td_\varphi(E)$ via r construction above

and in the case td_φ is invertible.

$$td_\varphi(T_X)^{-1} \cdot f_*^B(\varphi(a) \cdot td_\varphi(T_Y)) = \varphi(f_*^A(a))$$

or in standard form

$$f_*^B(\varphi(a) \cdot td_\varphi(T_Y)) = \varphi(f_*^A(a)) \cdot td_\varphi(T_X)$$

$$\begin{tikzcd} A(Y) \arrow[r, "td_\varphi(T_Y)\cdot\varphi"] \arrow[d, "f_*^A"'] \arrow[dr, phantom, "\circlearrowright"] & B(Y) \arrow[d, "f_*^B"] \\ A(X) \arrow[r, "td_\varphi(T_X)\varphi"'] & B(X) \end{tikzcd}$$

"R-R"

$A = K$ $\qquad$ $B = CH_{\mathbb{Q}}$

φ = chern character.

<u>Next stuff:</u>

A bit of classical K theory

K_0, K_1, K_2

$\rightsquigarrow$

Higher K-theory (following Quillen's Q construction)
roughly following Srinivas' Alg. K thy book.

[Fibrations, homotopy fibers, LES in hom gps.]

[Spectral sequences arising from exact couples]

<u>Classical K thy:</u> $\qquad$ $K^\cdot$ = contravariant functor

Given a scheme X

$K^\cdot X$ = free ab gp (iso classes loc. free shvs) / ses rel.

ring via $[E]\cdot[F] = [E \otimes F]$ $\qquad$ $[E] = [E'] + [E'']$ if $0 \to E' \to E \to E'' \to 0$

$$K_\cdot X = \text{free ab gp}\left(\begin{matrix}\text{iso}\\ \text{classes}\end{matrix} \text{ coherent shvs}\right) \Big/ \text{ses rel.}$$

$[E] = [E'] + [E'']$

if $0 \to E' \to E \to E'' \to 0$

<u>Variation:</u> Can do this for rings (not necessarily comm.)

$$K^\cdot(R) = \text{free ab gp gen by f.gen. proj. modules} \Big/ [E \oplus F] = [E] + [F]$$

$$K_\cdot(R) = \text{- - - - f. presented module} \Big/ \text{ses. relation.}$$

<u>ex:</u> F = field $\quad \underbrace{K(F) \cong \mathbb{Z}}_{\dim}$

$\mathbb{C}[x]/x^2+1$

$\mathbb{C}/\mathbb{R} \qquad K(\mathbb{C}) \longrightarrow K(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = K(\mathbb{C} \times \mathbb{C})$

$\mathbb{Z} \xrightarrow{\Delta} \; = \mathbb{Z} \times \mathbb{Z}$

$\mathbb{H}/\mathbb{R}$ $\qquad$ $M/\mathbb{H} \simeq \mathbb{H}^n$

$1,\ i, j\ ,\ k = ij$ $\qquad$ $K(\mathbb{H}) \simeq \mathbb{Z}_{\mathbb{H}}$

$i^2 = -1 = j^2\ ,\ ij = -ji$

$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$ simple s.s. alg

unique simple mod $\mathbb{C}^2$

$$K(M_2(\mathbb{C})) \simeq \mathbb{Z}_{\mathbb{C}^2}$$

$$\begin{array}{ccc} K(\mathbb{H}) & \longrightarrow & K(M_2(\mathbb{C})) \\ 1 & \longmapsto & 2 \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

$$\mathbb{H} \rightsquigarrow M_2(\mathbb{C}) = (\mathbb{C}^2)^2$$

$$K\left(\mathrm{Proj}\ \mathbb{R}[x,y,z]/x^2+y^2+z^2\right)$$ "Quillen's thm"

$$= K(\mathbb{R}) \times K(\mathbb{H})$$

Def $K_1(R)$ R ring.

$$GL_n(R) = (M_n(R))^*$$

$$GL_n(R) \hookrightarrow GL_{n+1}(R) \hookrightarrow \cdots \quad GL_\infty(R)$$

$$\begin{bmatrix} x \end{bmatrix} \quad \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_1(R) = GL_\infty(R) / [GL_\infty(R), GL_\infty(R)]$$

ex: $K_1(F) = F^*$
F field

$$K_1(\mathbb{H}) = (\mathbb{H}^*)^{ab} = \mathbb{R}^*$$

$$\begin{tikzcd} \mathbb{H} \arrow[r, hook] \arrow[rr, bend right] & M_2(\mathbb{C}) \arrow[r, "\det"] & \mathbb{C} \\ & & \mathbb{R} \arrow[u] \end{tikzcd}$$

more generally if A is a central simple k-Algebra

$$\begin{tikzcd} 0 \arrow[r] & SK_1(A) \arrow[r] & K_1(A) \arrow[r] & k^* \\ & (A^*)^{ab} \arrow[u, "\cong"] & A^* \arrow[u] \arrow[ur, "\mathrm{Nrd}"'] & \end{tikzcd}$$

A CSA $/_k$ $\quad \deg A = \sqrt{\dim_k A}$

if $\deg A$ is squarefree $\Rightarrow SK_1(A) = 0$

<u>Conj</u>: (Suslin) if A is not squarefree deg & division

then $\exists$ $L/_k$ s.t. $A \otimes_k L$ has $SK_1(A \otimes_k L) \neq 0$

proved if $\deg A = 4$

open in general p^2