Math 7240, Linear Algebraic Groups over Fields, Fall 2025
Homework

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from lecture 1

1. Let X be a k-space (that is, a functor from commutative k-algebras to sets). A field point of X is an element of $X(L)$ where L is a field extension of k. If $x \in X(L)$ and $y \in X(E)$ are field valued points of X, we say that $x \sim y$ if there exists a field extension M/k and morphisms of field extensions $\phi : E \to M$, $\psi : L \to M$ such that $X(\psi)(x) = X(\phi)(y)$ (in class we wrote this as $x_M = y_M$).

   Show that if $X = \operatorname{Spec} A$, then the equivalence classes of field valued points of X are in bijection with the prime ideals of A.

2. Recall that if X is a k-space which is of the form $X(R) = \mathrm{Hom}_{k-alg}(A, R)$ for some k-algebra A (i.e. X is a representable functor), we say that A is the coordinate ring of X and write $A = k[X]$. We also say in this case that X is the spectrum of A and write $X = \operatorname{Spec} A$. If X has this form, we say that X is an affine k-scheme.

   Show that Spec and $k[\ \_\ ]$ defines an equivalence of categories between the category of affine k-schemes (that is, the full subcategory of the category of k-spaces consisting of affine k-schemes) and the opposite of the category of k-algebras.

   note: you should prove this "from scratch," and not simply quote theorems from category theory!

3. (possibly challenging)

   (a) Show that if A, B are k-algebras, then $A \times B$ is the categorical product of A and B in the category of rings and $\operatorname{Spec}(A \times B)$ is the categorical coproduct of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ in the category of k-sheaves[1].

   (b) On the other hand, show that if $A_\lambda, \lambda \in \Lambda$ is a collection of nonzero k-algebras, then $\times_{\lambda \in \Lambda} A_\lambda$ is the categorical product of the A_λ's in the category of rings, but if Λ is infinite, $\operatorname{Spec}(\times_{\lambda \in \Lambda} A_\lambda)$ is <u>not</u> the categorical coproduct of the spaces $\operatorname{Spec} A_\lambda$ in the category of k-sheaves.

[1] note – this was incorrectly writen as spaces previously!

from lecture 2

4. Suppose $X = Z(f_1, \ldots, f_s)$ with $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ and $Y = Z(g_1, \ldots, g_t)$ with $g_1, \ldots, g_t \in k[y_1, \ldots, y_m]$ are finite type affine k-schemes. We think of these as sitting inside the affine spaces $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ where $\mathbb{A}^n$ has coordinate functions given by the x_i's and $\mathbb{A}^m$ has coordinate functions given by the y_i's.

   A morphism of affine schemes from X to Y is a collection of polynomials $\phi = (\phi_1, \ldots, \phi_m)$ with $\phi_i \in k[x_1, \ldots, x_n]$ (which we can think of as polynomial functions from $\mathbb{A}^n$ to $\mathbb{A}^m$), such that whenever we have an R-point of X, that is, $a = (a_1, \ldots, a_n) \in R^n$ such that $f_i(a) = 0$ for all i, we have $\phi(a) = (\phi_1(a), \phi_2(a), \ldots, \phi_m(a)) \in R^m$ is an R-point of Y.

   Show that morphisms of affine schemes $X \to Y$ are in bijection with natural transformations of functors from X to Y (considered as k-spaces).

5. Show that if the coordinate rings of X and Y are domains in the prior problem, if $\phi = (\phi_1, \ldots, \phi_m)$ with $\phi_i \in k[x_1, \ldots, x_n]$, then ϕ is a morphism from X to Y as affine schemes if and only if for every field extension L/k, we have an L-point of X, $a = (a_1, \ldots, a_n) \in L^n$ then $\phi(a) = (\phi_1(a), \phi_2(a), \ldots, \phi_m(a)) \in L^m$ is an L-point of Y.

6. (challenging) In the previous problem, show that instead of considering all field extensions L/k, it suffices to consider any single field extension L/k with L algebraically closed!

lecture 3

7. Consider the following functors from k-algebras to sets defined by

   1. $F : R \mapsto R[[t]]$
   2. $G : R \mapsto R((t)) = \{\sum_{i=d}^{\infty} a_i x^i \mid d \in \mathbb{Z}, a_i \in R\}$
   3. $H : R \mapsto R[[t]]^*$
   4. $K : R \mapsto R((t))^*$

   Which of these are representable? If a functor is representable, demonstrate it by finding a k-algebra representing it. If it isn't representable, provide an argument explaining why.

8. Consider the k-group space T_2, defined as the functor from rings to sets given by

$$T_2(R) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \; ad \in R^* \right\},$$

   with its usual group structure.

   (a) Show that T_2 is representable – that is, show that $T_2 = \mathrm{Spec}(A)$ for some k-algebra A.

   (b) Recall that this implies that A has a comultiplication operation $A \to A \otimes_k A$ corresponding to the group operation $T_2 \times T_2 \to T_2$. Give an explicit description of this comultiplication.

   (c) Similarly, there should be a map $S : A \to A$ (the "antipode") corresponding to group inversion $\iota : T_2 \to T_2$. Find an explicit description of the map S.

vaguely lecture 1-4 related, making connections to schemes

9. Let Top be the category of topological spaces. If F is a contravariant functor from Top to sets, we say that F is a Top-sheaf if for every topological space X, when F is restricted to open subsets of X, it forms a sheaf.

   Show that for every topological space Y, the functor $X \mapsto \mathrm{Hom}(X, Y)$ is a Top-sheaf.

10. Recall that a ringed space is a pair $X = (X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A morphism of locally ringed spaces $f = (f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a map of topological spaces $f : X \to Y$ together with a morphism of sheaves of rings $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

    (a) Fix a ringed space Y and consider the functor h_Y from ringed spaces to sets given by $h_Y(X) = Hom(X, Y)$. Show that if we restrict h_Y to the open subsets of X, we obtain a sheaf on X.

    (b) Recall that X is called locally ringed if the stalks $\mathcal{O}_{X,x}$ are local rings, and a morphism $f : X \to Y$ is a morphism of locally ringed spaces if the induced maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are *local* maps – that is, they take the maximal ideal of one to the maximal ideal of the other.

    Show that if Y is a locally ringed space and if we define a functor from locally ringed spaces to sets via $\widetilde{h}_Y(X) = Hom_{local}(X, Y)$, then again for every locally ringed space X, the restriction of $\widetilde{h}_Y$ to the open sets of X is a sheaf on X.

    (c) Use the above to show that if X is a scheme, then $R \mapsto Hom_{scheme}(\operatorname{Spec} R, X)$ is a k-sheaf.

11. Let X be a k-sheaf. Define a "relative power-set space" of X via

$$\mathcal{P}_X(R) = \{\text{subsheaves } F \subseteq X \times \operatorname{Spec} R\}$$

    with, for $\phi : R \to R'$, we define

$$\mathcal{P}_X(\phi)(F) = F \times_{\operatorname{Spec} R} \operatorname{Spec} R' \subseteq X \times \operatorname{Spec} R' \cong (X \times \operatorname{Spec} R) \times_{\operatorname{Spec} R} \operatorname{Spec} R'.$$

    Show that $\mathcal{P}_X$ is a k-sheaf.

Lecture 11

12. (projective space) Let $\mathbb{P}^n$ denote that k-scheme obtained by gluing the $n+1$ copies of affine n space $\mathbb{A}^n_i$, $i = 0, \ldots, n$, where we regard $\mathbb{A}^n_i = \operatorname{Spec}(k[x_0/x_i, \cdots, x_n/x_i])$, glued together via

$$\mathbb{A}^n_{i,j} = \operatorname{Spec}(k[x_0/x_i, \ldots, x_n/x_i, x_i/x_j]) = \operatorname{Spec}(k[x_0/x_j, \ldots x_n/x_j, x_j/x_i]) \subseteq \mathbb{A}^n_i, \mathbb{A}^n_j.$$

    Let $\widetilde{\mathbb{P}}^n$ denote the k-space given by

$$\widetilde{\mathbb{P}}^n(R) = \{L \subseteq R^n \mid L \text{ is a rank 1 projective } R\text{-modules, and } \exists Q, Q \otimes L \cong R^n\},$$

    and for $R \to S$, we obtain $\mathbb{P}^n(R) \to \mathbb{P}^n(S)$ by sending $L \subseteq R^n$ to $L \otimes S \subseteq R^n \otimes_R S = S^n$.

    (a) Show that $\widetilde{\mathbb{P}}^n$ is a k-sheaf.

    (b) Show that the maps (for example for $i = 1$ given by) $\mathbb{A}^n_1 \to \widetilde{\mathbb{P}}^n$ given by sending $(a_1, \ldots, a_n)$ to the inclusion $R \hookrightarrow R^n$ via $r \mapsto (1, a_1, \ldots, a_n)$ (more generally for $\mathbb{A}^n_i$ giving a 1 in the i'th place) determine morphisms which agree on the $\mathbb{A}^n_{i,j}$ and hence (using Exercise 14) induce morphisms of k-spaces $\phi : \mathbb{P}^n \to \widetilde{\mathbb{P}}^n$.

    (c) Show that ϕ is bijective for R a local ring. Conclude that ϕ is an isomorphism of k-sheaves.

Lecture 12

13. Let X, Y be k-sheaves and suppose we have $j_X : X \to V$, $j_Y : Y \to V$ open inclusions. Show that the fiber product of k-sheaves if given via $X \times_V Y(R) = X(R) \cap Y(R)$ where the intersection is taken via the inclusions $j_X(R) : X(R) \to V(R)$ and $j_Y(R) : Y(R) \to V(R)$. (why is this again a sheaf?).

    Conclude that the functor taking k-schemes to k-sheaves preserves intersections of open inclusions.

14. Let X, Y be k-sheaves and suppose that we have $i : U \to X, j : U \to Y$ open inclusions. Show that the pushout $X \sqcup_U Y$ is given as follows: Def: a broken R-point of X and Y over U is a collection of open subsets $V_U \subseteq V_X, V_Y \subseteq Spec(R)$ and maps $f_X : V_X \to X$, $f_Y : V_Y \to Y$ such that $f_X|_{V_U}, f_Y|_{V_U} : V_U \to U$ agree (i.e. both maps have images landing in U and these maps agree). Given broken R-points $f = (V_U, V_X, V_Y, f_X, f_Y), f' = (V'_U, V'_X, V'_Y, f'_X, f'_Y)$ we say that $f \sim f'$ if $f_X|_{V_X \cap V'_X} = f'_X|_{V_X \cap V'_X}$ and similarly for the f_Y, f'_Y. Show that $X \sqcup_U Y$ represents the gluing of the schemes X and Y over U (as in Hartshorne II.2.3.5).

15. Use the previous exercise to conclude that the natural functor from k-schemes to k-sheaves preserves the operation of gluing as in Hartshorne, Exercise II.2.12.

16. Define a k-schematic sheaf to be a contravariant functor X from k-schemes to the category of sets such that for any k-scheme Y, the restriction of X to the open subschemes of Y yields a sheaf. Let sSh_k be the category of k-schematic sheaves, where morphisms are given by natural transformations of functors. Let Sh_k be the category of k-sheaves.

    We note that, by identifying the category of k-algebras with the opposite of the category of affine k-schemes, we obtain a functor $sSh_k \to Sh_k$ (via restriction the domain of the functor to affine schemes). Show that this is an equivalence of categories.

17. Let X, Y be k-schemes and let $Z \subseteq X \times Y$ be a closed k-subscheme. Recall that we have defined $\cap_Y Z$ to be the subscheme of X whose R-points $\cap_Y Z(R)$ are exactly those R points $x \in X(R)$ such that the natural map of schemes $x \times \mathrm{id}_Y : \operatorname{Spec} R \times Y \to X \times Y$ factors through the inclusion $Z \to X \times Y$.

    (a) Show that if $Y = \bigcup Y_i$ is a union of open subschemes, then $\cap_Y Z = \bigcap_i (\cap_{Y_i} Z)$. Note that you may think of the intersection either in k-schemes or k-spaces by Problem 13.

    Conclude that if $\cap_{Y_i} Z|_{X \times Y_i}$ is closed in $X \times Y_i$ for each i, then $\cap_Y Z$ is closed in $X \times Y$.

    (b) Show that if $X = \bigcup X_i$ is a union of open subschemes, then $\bigcup_i (\cap_Y Z|_{X_i \times Y}) = \cap_Y Z$. Note that you may think of the union either in k-schemes or in k-spaces by Problem 14.

    Conclude that if $\cap_Y Z|_{X_i \times Y}$ is closed in $X_i \times Y$ for each i, then $\cap_Y Z$ is closed in $X \times Y$.