Linear alg gps over fields

Familiarity w/ alg geom (varieties, hopefully schemes)

First 2 weeks (4 lectures) review of alg geom.

12 problems by end (not at end) for A.

Weekly "coffee hour" mondays after class until 11:30

or maybe later (after lunch)

or wed?

or something TBD

Humphrey's LAG & notes of Brian Conrad.

Affine varieties & schemes (today rings are commutative!)

Idea: affine scheme (over a field) is defined as the "zeros" of a collection of polynomial equations.

$Z(x^2-y)$ in variables x, y over field k

$Z(x^2+y^2)$ over $\mathbb{R}$

given some k-alg $R \rightsquigarrow$ set of solns in R

$Z = Z(x^2+y^2)$ $\quad Z(\mathbb{R}) = \{(a,b) \in \mathbb{R}^2 \mid a^2+b^2=0\}$
$= \{(0,0)\}$

$Z(\mathbb{C})$ = lots of solns $\{(a, \pm i a) \mid a \in \mathbb{C}\}$

$Z(\mathbb{C}[\varepsilon]/\varepsilon^2) = \{(a+b\varepsilon, \pm i \quad \}$

<u>Def</u> (temporary) let k be a field. A k-space is a functor from k-algebras to sets.

X a space, R a k-algebra $\quad X(R)$ = set of R-valued points of X.

<u>Def</u> let k be a field, An affine k-scheme is a k-space of form $Z(f_1, \dots, f_m)$ f_i polys in some # of vars.

<u>Def</u> if $f_1, \dots, f_m \in k[x_1, \dots, x_n]$, define $Z(f_1, \dots, f_m) = Z$ to be the k-space given by

$$Z(R) = \{ r = (r_1, \dots, r_n) \in R^n \mid f_i(r) = 0 \text{ all } i \}$$

<u>Observation</u>:

If we let n, m be infinite above (∞ly many eqns & vars)

then affine schemes = representable functors.

i.e. $Z(f_1, \dots, f_m)(R) = \{(r_1, \dots, r_n) \in R^n \mid f_i(\vec{r}) = 0\}$

$$(r_1, \dots, r_n) \in R^n \longleftrightarrow \underset{k\text{-alg}}{\mathrm{Hom}}(k[x_1, \dots, x_n], R) \ni \varphi$$

$$f(r_1, \dots, r_n) = 0 \longleftrightarrow \varphi(f) = 0$$

$$\underset{☆}{Z(f_1, \dots, f_m)(R)} = \mathrm{Hom}_{k\text{-alg}}\left(\frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_m)}, R\right)$$

$$\cap\!\!|$$

$$R^n = \mathrm{Hom}_{k\text{-alg}}(k[x_1, \dots, x_n], R)$$

<u>Def</u> if $F: \mathcal{C} \to \mathcal{D}$ functor, we say F is rep. by some $c \in \mathcal{C}$ if $\exists$ nat $\cong$ $F \cong \mathrm{Hom}(c, -)$

<u>Note</u>: could just say $Z(I)$ $I = (f_1, \dots, f_m)$ since Z only depends on ideal by ☆

We like fields, and are particularly interested in $X(L)$ for L a field.

<u>Def</u> if X is a k-space, E, L are field exts of k and $x \in X(E)$, $y \in X(L)$ we say $x \sim y$ if $\exists$ field ext. M and imbeddings $E, L \hookrightarrow M$

s.t. $x_M = y_M$

Notation: X a space, $R \xrightarrow{\varphi} S$ is a hom of k-algs

$$X(R) \xrightarrow{X(\varphi)} X(S)$$
$$x \longmapsto X(\varphi)(x) \equiv x_S \text{ or } \varphi(x)$$

for $Z = Z(f_1, \dots, f_m)$ $\quad Z(R) \longrightarrow Z(S)$

$$(r_1, \dots, r_n) \longmapsto (\varphi(r_1), \dots, \varphi(r_n))$$

ex: $Z(x^2+1)/\mathbb{R} = \varnothing$

$Z(\mathbb{C}) = \{\pm i\}$ $\qquad i \in \mathbb{C} \xrightarrow{id} \mathbb{C}$

$\qquad\qquad -i \in \mathbb{C} \xrightarrow{-} $

Exercise: Field valued pts of $Z(I)$ up to equivalence are in bijection w/ prime ideals of $k[x_1, \dots, x_n]/I$.

Def scheme theoretic pts of $X \equiv$ Field pts $/\sim$.

Affine schemes $\longleftrightarrow$ rep functors (w/ only many variables allowed)

$$Z \longleftrightarrow \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_m)} = k[Z]$$

$Z(f_1, \dots, f_m)$ notation

(affine) coordinate ring of Z

Ref Spec A $\longleftrightarrow$ A (some k-algebra)

affine scheme corresponding to A

Def we say that Z has <u>finite type</u> if $k[Z]$ is finitely generated

<u>Procedure:</u>

$$k\text{-alg}: A = \frac{k[x_1, \dots]}{I} = \frac{k[x_a \mid a \in A]}{\ker \varphi}$$

$$k[x_a] \xrightarrow{\varphi} A$$
$$x_a \longmapsto a$$

$$\frac{k[x_a]}{\ker \varphi} \rightsquigarrow Z(f_\lambda \mid \lambda \in \Lambda)$$

$\ker \varphi = (f_\lambda \mid \lambda \in \Lambda)$

$$Z(R) = \{(r_a) \in R^{|A|} \mid f_\lambda(\vec{r}) = 0\}$$

Products

if $\mathcal{C}$ a cat. $a,b \in \mathcal{C}$ define $a \times b$

to be an obj w/ morphisms $a\times b \to a$, $a \times b \to b$

s.t. if $x \to a$, $x \to b$ $\exists !\ x \to a \times b$

s.t.
$$\begin{tikzcd} x \arrow[r] \arrow[rr, bend left] \arrow[rrd, bend right] & a\times b \arrow[r] \arrow[rd] & a \\ & & b \end{tikzcd}$$
commutes.

dual notion: coproduct
or sum

$$\begin{tikzcd} a \arrow[r] & a \oplus b \text{ or } a \amalg b \\ b \arrow[ru] & \end{tikzcd}$$

etc.

affine k-schemes $\longleftrightarrow$ k-algebras

$X \longleftrightarrow k[X]$

$\operatorname{Spec} A \longleftrightarrow A$

observation: if $A \xrightarrow{\varphi} B$ of k-algs

get $\operatorname{Spec} B \to \operatorname{Spec} A$

$\operatorname{Spec} B(R) \longrightarrow \operatorname{Spec} A(R)$

$\quad\ \ \| \qquad\qquad\qquad\quad \|$

$$\mathrm{Hom}(B,R) \qquad\qquad \mathrm{Hom}(A,R)$$

$$B \to R \qquad\qquad A \xrightarrow{\varphi} B \to R$$

conversely $\quad \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A \;$ gives $\; A \to B$

$$\mathrm{Spec}\, B(B) \longrightarrow \mathrm{Spec}\, A(B)$$

$$\begin{array}{ccc} \| & & \| \\ \mathrm{Hom}_{k\text{-alg}}(B,B) & & \mathrm{Hom}(A,B) \\ \Downarrow & & \\ \mathrm{id}_B & & \end{array}$$

<u>Exercise:</u> show these give inverse bijections

$$\mathrm{Hom}_{k\text{-sch}}(\mathrm{Spec}\, B, \mathrm{Spec}\, A) = \mathrm{Hom}_{k\text{-alg}}(A,B)$$

In other words $\quad$ Affine k-schemes $\underset{eq}{\cong} (k\text{-algs})^{op}$

Products & coproducts in k-algs?

$A, B \qquad A \times B$ is the categorical product

$\qquad\quad\; A \otimes_k B$ is the categorical coproduct.

$$\mathrm{Spec}\, A \times B = \mathrm{Spec}\, A \amalg \mathrm{Spec}\, B$$

$$\mathrm{Spec}\, A \otimes_k B = \mathrm{Spec}\, A \times_k \mathrm{Spec}\, B$$

Natural question: how do the scheme theoretic pts of $\operatorname{Spec} A \amalg \operatorname{Spec} B$ and $\operatorname{Spec} A \times_k \operatorname{Spec} B$ relate to those of $\operatorname{Spec} A$ & $\operatorname{Spec} B$?

& how do sets of R-pts relate?