Smoothness & connectedness of Alg. gps

$\rightsquigarrow$ $G$ smooth & affine gp variety $/k$ $\Rightarrow$ $G$ a "linear alg gp"

i.e. $G$ closed subgp-scheme of $GL_n$

Important input: $G$ admits a faithful f.d. rep.

Strategy: $G$ "acts on" $k[G]$

($k[G]$ is a $k[G]$-comodule)

Want: f.dim'l subspace $W \subseteq k[G]$ which is a (faithful) subcomodule.

At some point in the past we using following strategy:

If $G$ acts on an affine scheme $X$

i.e. $G \times X \xrightarrow{a} X$ s.t.

$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times id_X"] \arrow[d, "id_G \times a"'] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"'] & X \end{tikzcd}$$

$$\begin{tikzcd} X = pt \times X \arrow[r, "e \times id_X"] \arrow[dr, bend right, "id_X"'] & G \times X \arrow[d, "m"] \\ & X \end{tikzcd}$$

then get $k[X]$ is a $k[G]$-comodule

$$k[X] \xrightarrow{a^\#} k[G] \otimes k[X]$$

$$k[G] \xrightarrow{m^\#} k[G] \otimes k[G]$$

**and:** if $w \in k[X]$ will show $w \subseteq W$ a f.d. $k[G]$-submodule.

write $a^\#(w) = \sum_i f_i \otimes w_i$ $\quad W = \langle w_i \rangle$

"last" time showed $w \in W$ (used unit axiom)

Let $\{f_i\}$ be an (infinite) basis for $k[G]$

$$(1 \otimes a^\#)(a^\# w) = \sum f_i \otimes a^\#(w_i)$$

$$\parallel$$

**WTS:** $a^\#(w_i) \subseteq k[G] \otimes W$

$$(m^\# \otimes 1)(a^\# w)$$

$$= \sum_i m^\#(f_i) \otimes w_i = \sum_{i,j} f_j \otimes \varphi_{ij} \otimes w_i$$

$$m^\#(f_i) = \sum_j f_j \otimes \varphi_{ij}$$

$f_j$ basis for $k[G]$

$\varphi_{ij} \in k[G]$

$$\sum f_i \otimes a^\#(w_i) = \sum_{j,i} f_i \otimes \varphi_{ji} \otimes w_j$$

$$= \sum_i f_i \otimes \left(\sum_j \varphi_{ji} \otimes w_j\right)$$

$$a^\#(w_i) = \sum_j \varphi_{ji} \otimes w_j$$

$\in k[G] \otimes W$. $\Delta$.

## Back to smoothness

**Last time**: talked about regularity & formal smoothness

- $(R, m)$ a local ring is regular if $R$ is Noeth &
  (Krull) $\dim R = \dim_{R/m} m/m^2$.

- $R$ is formally smooth over $k$ ($k$ can be a gen. comm. ring)
  if for any $S$ w/ ideal $J \lhd S$ s.t. $J^2 = 0$ and
  a comm. diagram then $\exists\ \operatorname{Spec} S \to \operatorname{Spec} R$

$$\begin{tikzcd} \operatorname{Spec} S/J \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} k \end{tikzcd} \quad \text{s.t.} \quad \sim \begin{tikzcd} \operatorname{Spec} S/J \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} S \arrow[r] \arrow[ur] & \operatorname{Spec} k \end{tikzcd}$$

commutes.

- $R$ is formally étale over $k$ ($k$ can be a gen. comm. ring)
  if for any $S$ w/ ideal $J \lhd S$ s.t. $J^2 = 0$ and
  a comm. diagram then $\exists!\ \operatorname{Spec} S \to \operatorname{Spec} R$

$$\begin{tikzcd} \operatorname{Spec} S/J \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} k \end{tikzcd} \quad \text{s.t.} \quad \sim \begin{tikzcd} \operatorname{Spec} S/J \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} S \arrow[r] \arrow[ur] & \operatorname{Spec} k \end{tikzcd}$$

commutes.

- $R$ is formally unramified over $k$ ($k$ can be a gen. comm. ring)

if for any $S$ w/ ideal $J \triangleleft S$ s.t. $J^2 = 0$ and a comm. diagram

$$\begin{tikzcd} \operatorname{Spec} S/J \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

then $\exists$ at most 1 $\operatorname{Spec} S \to \operatorname{Spec} R$ s.t.

$$\begin{tikzcd} \operatorname{Spec} S/J \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} S \arrow[r] \arrow[ur] & \operatorname{Spec} k \end{tikzcd}$$

commutes.

<u>Facts</u> variety $X/k$ is smooth $\iff X_{\bar{k}}$ is regular

$\iff X_{\bar{k}}$ is smooth

<u>Def</u> A morphism $f: X \to Y$ of varieties is <u>smooth</u> if for each $U \subset Y$ affine, $V \subset X$ affine $V \subseteq f^{-1}U$,

$V \to U$ formally smooth.

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<u>Last time</u>: showed $GL_n, SL_n$ (formally) smooth $/R$

via lifting matrices in $SL_n(S/J) \leftarrow SL_n(S)$

**Prop** (8.1.2 Conrad's notes) $G/k$ smooth.

- If $G$ acts on a scheme $X$, $X/k$ smooth, geom. connected ($X_{\bar{k}}$ is connected)
- If $G(\bar{k})$ acts transitively on $X(\bar{k})$
- If $\mathrm{Stab}_{G(\bar{k})}(x_0)$ $x_0 \in X(\bar{k})$ is connected $= (e)$

then $G$ is connected.

**Pf:** Enough to show $G_{\bar{k}}$ is connected so WLOG $k = \bar{k}$

Choose $x_0 \in X(k)$ $G$ regular, $G \twoheadrightarrow X$
$g \longmapsto g \cdot x_0$

fibers are all conjugate so $\simeq$ as schemes, same dimension.

("miracle flatness theorem" says if $V \to W$ w/ $V$ regular surjective. $W$ Cohen-Macaulay, fibers all same dim then flat.)

$\Rightarrow G \to X$ is flat hence open.

[ top $\Rightarrow$ open cont. surjection to connected space w/ connected fibers has connected domain ]

$\square$.

## Connectedness examples

$k[GL_n] = k[x_{ij}]_{i,j\in\{1,\dots,n\}}[\det(x_{ij})^{-1}]$ open in $\mathbb{A}^{n^2}$ so connected.

$k[SL_n] = k[x_{ij}] \dots / \langle \det(x_{ij}) - 1 \rangle$

$SL_n \circlearrowright X$

## Proj. space

given $V/k$ v.space, $\mathbb{P}(V)$

functor $\underline{k\text{-alg}} \xrightarrow{\mathbb{P}(V)} \underline{\text{Sets}}$

$R \longmapsto \mathbb{P}^{\vee}(V)(R) = \{\text{quotients } V\otimes_k R \to L\}$ where $L/R$ is rank 1 projective module.

$\mathbb{P}(V)(R) = \{\text{sub modules } L \subseteq V\otimes_k R$ s.t. $L/R$ rk 1 projective & $\exists Q$ w/ $L\oplus Q \cong V\otimes_k R\}$

$R = k[x]$

$L \subseteq V\otimes_k k[x]$

$\}x=a$

$L\otimes_{k[x]} k[x]/x-a \subseteq V\otimes_k k[x]/x-a \rightsquigarrow L_a \subseteq V$

inclusion $\Leftrightarrow \exists Q$.

Similarly: $V/k$ v.space

$Gr(m,V)$

defined by $Gr(m,V)(R) = \left\{ P \subseteq V \otimes_k R \text{, } P \text{ proj rk } m \text{ \& } \exists\, Q \subseteq V \otimes_k R \text{ w/ } P \oplus Q \simeq V \otimes_k R \right\}$

$$Gr(m,V)^\vee(R) = \left\{ \text{surjections } V \otimes_k R \to P,\ P \text{ rk } m \text{ proj} \right\}$$

Fact: $Gr(m,V)^\vee = Gr(\dim V - m, V)$

$(V \otimes R \xrightarrow{\varphi} P) \rightsquigarrow \ker \varphi \subseteq V \otimes_k R$

$\mathbb{P}^\vee(V) = Gr(\dim V - 1, V)$

Flags: $\dim V = n$

$$Fl(V)(R) = \left\{ P_1 \subset P_2 \subset \dots \subset P_{n-1} \subset V \otimes_k R \mid P_i \in Gr(i,V) \right\}$$