Projective space:

two definitions: $\mathbb{P}^n$ obtained by gluing $n+1$ affine spaces

$$\mathbb{A}^n_0, \mathbb{A}^n_1, \ldots$$

$$\mathbb{A}^n_0 = \operatorname{Spec} k[\underbrace{x_1/x_0}_{u_1}, \underbrace{x_2/x_0}_{u_2} \ldots] \qquad \mathbb{A}^n_1 = \operatorname{Spec} k[\underbrace{x_0/x_1}_{v_0}, \underbrace{x_2/x_1}_{v_2} \ldots]$$

$$\operatorname{Spec} \ k[u_1, u_1^{-1}, u_2 \ldots] = k[v_0, v_0^{-1}, v_2 \ldots]$$

$$\tilde{\mathbb{P}}^n(R) = \{ L \subseteq R^{n+1} \text{ proj. rank 1 s.t. } \exists Q \text{ s.t. } L \oplus Q \simeq R^{n+1} \}$$

$$\begin{tikzcd}
 & \mathbb{A}^n_0 \arrow[dr] \arrow[drr, bend left] & & \\
\mathbb{A}^n_0 \cap \mathbb{A}^n_1 \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{P}^n \arrow[r, dashed] & \tilde{\mathbb{P}}^n \\
 & \mathbb{A}^n_1 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

$$0 \to R \xrightarrow{a \mapsto (a, ar_1, \ldots)} R^n \to Q \to 0$$

$R^n \to R$: 1^{st} coord.

$$\mathbb{A}^n_0(R) \to \tilde{\mathbb{P}}^n(R)$$

$$(r_1, \ldots, r_n) \longmapsto \underbrace{R \hookrightarrow R^n}_{\text{"}L\text{"}}$$

"$[1; r_1; \ldots; r_n]$" $=$ line $\{(a, ar_1, \ldots, ar_n) \mid a \in R\}$

to show isom = bijection for each R

for each R, comparing sheaves on $\operatorname{Spec} R$ w/ Zariski top!

Ex: $\tilde{\mathbb{P}}^n$ is a Zsheaf.

Ex: show we really get a morphism $\mathbb{P}^n \to \widetilde{\mathbb{P}}^n$ as above.

to show iso, enough to check when R = local rings (stalks)

R local $\Rightarrow L \cong R \ldots$

Ex: check this is bijective for local rings.

amounts to showing if $R \hookrightarrow R^{n+1}$

$r \mapsto (ra_0, ra_1, \ldots, ra_n)$

then injection is split $\iff a_i \in R^*$ some i

(R local!)

Grassmannians

$Gr(m, V)$ (m-dim'l subspaces of a v.space V)

$\hookrightarrow \mathbb{P}(\Lambda^m V)$

$W \subset V \otimes_k R$ rk m proj. summand

$L \subseteq$

$W \longmapsto \Lambda^m W \subseteq \Lambda^m V$

$W \in Gr(m, V)$

rk 1 proj summand.

$\Lambda^m W$ are those $L \subseteq \Lambda^m V$ s.t.

for all $\omega \in \Lambda^{m-1} V^*$, $L \wedge (\omega \lrcorner L) = 0$

cut out $Gr(m,V)$ in $\mathbb{P}(\Lambda^m V)$

$$(\Lambda^r V^*) \times (\Lambda^m V) \to \Lambda^{m-r} V$$

$$Fl_{m_1,\dots,m_e}(V)(R) = \left\{ W_1 \subseteq \dots \subseteq W_e \subseteq V\otimes_k R \;\middle|\; rk\, W_i = m_i \;\&\; [W_i] \in Gr(m_i, V) \right\}$$

$0 < m_1 < m_2 < \dots < n = \dim V$

$$Fl(V) \equiv Fl_{1,2,\dots,n-1}(V)$$

$$SL_n(R) \circlearrowright Fl(V)(R)$$

$\ni T$ lin. trans on $V\otimes_k R$ $\circlearrowright W \subseteq V \otimes_k R$

at $R = \bar{k}$ $SL_n(\bar{k})$ acts transitively $(W_1 \subset \dots \subseteq W_{n-1})$ defined by ordered basis $e_1,\dots,e_n$ (w/ det 1)

$W_i = \langle e_1,\dots,e_i\rangle$

<u>Stabilizer</u>:

$W_i = \langle e_1,\dots,e_i\rangle$ standard basis

$$\begin{bmatrix} * & * & & \\ 0 & * & \ddots & * \\ \vdots & 0 & & \\ \vdots & \vdots & \ddots & \\ 0 & \vdots & 0 & * \end{bmatrix} = \begin{bmatrix} a_1 & & & * \\ & \ddots & & \\ & & a_{n-1} & \\ 0 & & & (\prod a_i)^{-1} \end{bmatrix}$$

$$\text{Stab} \simeq \mathbb{A}^{\frac{n^2+n}{2}-1} \quad \underline{\text{connected}}.$$

<u>Ex</u>: $Fl(V)$ smooth

$\Rightarrow$ Prop last fre SL_n connected.

Closed orbit lemma

Lemma Let G be a smooth k-gp variety, $G \curvearrowright X$ action on a finite type scheme/k. if $x \in X(k)$ point can consider (for any R) orbit $G(R) \cdot x \subseteq X(R)$

Looking at $R = \bar{k}$, then the orbit $G(\bar{k}) \cdot x$ is locally closed in X and if $G(\bar{k}) \cdot x$ has min'l dimension (over all orbits) then it is closed.

$\mathbb{G}_m \curvearrowright \mathbb{A}^1$ $\qquad$ $\mathbb{G}_m(\bar{k}) \cdot x = \mathbb{A}^1 \setminus \{0\}$ if $x \neq 0$

$\lambda \cdot x = \lambda x$ $\qquad$ $\mathbb{G}_m \cdot 0 = \{0\}$

Pf (sketch)

work over $k = \bar{k}$

first, observe that if $Y = \overline{G(k) \cdot x}$ then G acts on Y as well. $G \cdot Y \subseteq Y$.

So enough to show $G(k) \cdot x \subseteq Y$ open.

Have a map $G(k) \longrightarrow Y$ morphism $\Rightarrow$ image $(G(k) \cdot x)$ is constructible

$g \longmapsto g \cdot x$

$\Rightarrow G(k)\cdot x$ locally closed in Y

closed inside of open but also dense in Y.

So image $G(k)\cdot x$ must contain a nonempty open $U \ni x_0$ in Y,

$\Rightarrow$ image is open since $G(k)\cdot x = \bigcup_{g \in G(k)} g\cdot U \supseteq \{g\cdot x_0\} = G(k)\cdot x$.

$\Rightarrow G(k)\cdot x$ open in $Y \underset{\text{closed}}{\subseteq} X$ $\Rightarrow$ orbit is loc. closed.

$G(k)\cdot x \overset{\text{open}}{\subseteq} Y$

If dim $G(k)\cdot x$ minimal over orbits then can't have $y \in Y \setminus G(k)\cdot x$

since $G(k)\cdot y$ would be loc. closed in strict complement of open $G(k)\cdot x$

would have smaller dim.

So $G(k)\cdot x$ min'l dim $\Rightarrow$ closed.

$\square$