1. G smooth affine gp variety$/k$ $\Rightarrow G \hookrightarrow GL_n$ some n as a closed subgp.

2. Centralizers, Normalizers, Transporters

3. Chevalley-Jordan decomposition

<u>Last time</u>:

closed orbit lemma:

if $G \curvearrowright X$ G smooth k-gp variety then orbits are locally closed & orbits of min'l dimension are closed.

<u>Lemma</u>:

Let $f: G \to G'$ a morphism of sm. affine gp varieties. then image of f in G' is closed.

<u>Pf</u>: f gives rise to an action of G on G' & all orbits have same dim. (conjugate via right mult in G')

$\Rightarrow$ all min'l $\Rightarrow$ all closed. $\square$

Lemma (compare Conrad 11.1.2) $\boxed{\text{FALSE!}}$

Suppose $f : X \to Y$ map of affine varieties over a field k s.t. $\forall\, L/k$ field exts w/ $L = \bar{L}$, $X(L) \to Y(L)$ is bijective.

Then f is an iso.

Pf: $X = \operatorname{Spec} B$ $\quad Y = \operatorname{Spec} A$ $\quad f^* : A \to B$

Let $L = \overline{\operatorname{frac}(A)}$. Have L-pt of Y given by $A \hookrightarrow L$

Hyp $\Rightarrow \exists\; B \to L$ s.t.

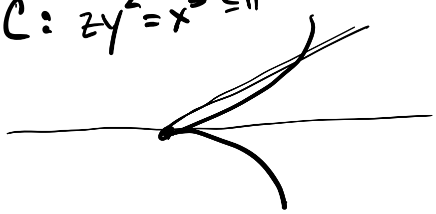
$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend right, "\text{inclusion}"'] & B \arrow[r] & L \end{tikzcd}$$

is above inclusion

Let $I = \ker(B \to L)$ $\qquad A \hookrightarrow B$

$A \cap I = (0)$. $\qquad A \oplus I = B$

$C : zy^2 = x^3 \subseteq \mathbb{P}^2$



$\mathbb{P}^1 \longrightarrow C$

$$\begin{tikzcd} \mathbb{C}[x^2, x^3] \arrow[r, phantom, "\subseteq"] \arrow[dr, "\varphi"'] & \mathbb{C}[x] \arrow[d, dashed, "\psi"] \\ & L \end{tikzcd}$$

$\varphi(x^2) = 0$ or $\varphi(x^3) = 0$

$\psi(x) = 0$

else $\psi(x) = \varphi(x^3) / \varphi(x^2)$

Want: if $G \xrightarrow{f} G'$ is a scheme-theoretically surj. morphism of k-gp varieties w/ trivial scheme-theoretic kernel $\Rightarrow$ f is an iso.

$$\left(\ker f : \begin{array}{ccc} \ker f & \to & G \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k & \xrightarrow[e]{} & G' \end{array} \right)$$

Pf. strategy:

$\ker f = (\operatorname{Spec} k \to G) \Rightarrow f$ is bijective on field pts all L/k, $L = \bar{L}$.

if $x \in G'(k)$ k-pt, s.t. $x = f(y)$, $y \in G$, $f^{-1}x \simeq \ker f$ via translation by y or y^{-1}

But scheme theoretic $\ker = \operatorname{Spec} k \Rightarrow \ker f_L \simeq \operatorname{Spec} L$

$\square$ after lemma...

☆ **Lem** (Conrad 11.1.1) gp varieties/k
$f : G \to G'$ has scheme-theoretically trivial kernel $\Leftrightarrow$ $f(L)$ has trivial kernel all $L = \bar{L}$ L/k, and Tef has trivial kernel.

Brief tangent space digression

$X = \operatorname{Spec} R \qquad x \in X \Leftrightarrow m_x \lhd R$ max'l ideal

then $T_x X \equiv \left(m_x / m_x^2 \right)^*$

if $x \in X(k)$ $\quad x \in \operatorname{Hom}(\operatorname{Spec} k, \operatorname{Spec} R) \qquad R \to k$

$\uparrow \pi \qquad\qquad R \to k(\varepsilon)/\varepsilon^2$?

$\tilde{x} \in \operatorname{Hom}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, \operatorname{Spec} R)$

Given $\tilde{x}$ s.t. $\tilde{x} \mapsto x \qquad R_{m_x} \to k[\varepsilon]/\varepsilon^2$

$$\begin{array}{c} \downarrow \\ k \end{array}$$

x's

$\boxed{\pi^{-1}(x) = T_x X}$

$m_x \rightsquigarrow \varepsilon\, k[\varepsilon]/\varepsilon^2 = \varepsilon k \simeq k$

$m_x^2 \rightsquigarrow \left(\varepsilon\, k[\varepsilon]/\varepsilon^2\right)^2 = 0$

$\tilde{x}|_{m_x} : m_x/m_x^2 \longrightarrow \varepsilon k \simeq k$

k-lin map.

$\mathbb{T} = \operatorname{Spec} k[\varepsilon]/\varepsilon^2$

"ring of dual numbers"

$(g, v) \longmapsto g = e$

$$0 \to T_e G \to G(k[\varepsilon]/\varepsilon^2) \to G(k)$$

<u>exercise:</u> $T_eG \cong (m_e/m_e^2)^*$ as groups (additive)

$G \xrightarrow{f} G'$ gp varieties$/k$

$H = \ker f$ (scheme-theoretic)

$$\begin{tikzcd}
\mathrm{Spec}\,R \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr] & \ker f \arrow[r] \arrow[d] & G \arrow[d] \arrow[d, bend left] \\
& \mathrm{Spec}\,k \arrow[r, "e"] & G'
\end{tikzcd}$$

$e_R \in G'(R)$

$e \in G'(k)$

$\mathrm{Spec}\,R \to \mathrm{Spec}\,k \xrightarrow{e} G'$

$\ker(G(k) \to G'(k)) = \ker f(k)$

$\ker(G(R) \to G'(R)) = \ker f(R)$

$$\begin{tikzcd}
0 \arrow[r] & H(k) \arrow[r] & G(k) \arrow[r, "f(k)"] & G'(k) \\
0 \arrow[r] & H(k[\varepsilon]/\varepsilon^2) \arrow[r] \arrow[u] & G(k[\varepsilon]/\varepsilon^2) \arrow[r, "f"] \arrow[u] & G'(k[\varepsilon]/\varepsilon^2) \arrow[u] \\
0 \arrow[r] & T_eH \arrow[r] \arrow[u] & T_eG \arrow[r, "T_ef"] \arrow[u] & T_eG' \arrow[u] \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

$\Rightarrow \ker(T_e f) = T_e(\ker f)$

So $\ker f = \operatorname{Spec} A$ has unique prime ideal (local Artin ring)

$(A, m) \quad (m/m^2)^* = 0 \quad \Rightarrow m = m^2$

A Noeth. $\Rightarrow m = 0 \Rightarrow A =$ field.

$A = \operatorname{Spec} L \quad L/k$ field ext. $\quad e \in (\ker f)(k)$

$$\Rightarrow \exists \text{ a } k\text{-map } \begin{tikzcd} L \arrow[r] & k \\ k \arrow[u, dash] & \end{tikzcd} \Rightarrow L = k.$$

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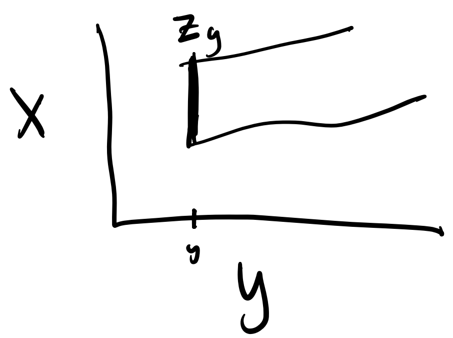
given "family of closed sets in X parametrized by Y"



$Z \subseteq X \times Y$ closed

$\bigcap_{y \in Y} Z_y$

Next time: Def: $\bigcap_Y Z \subseteq X$ closed.