Prop: Let $f: G \to H$ be a morphism of smooth gp varieties s.t. $G(L) \to H(L)$ is injective for L/k, $L = \bar{L}$ & s.t. $T_e f$ is injective. Then the induced map $G \to f(G) \subseteq H$ is an iso of G w/ a closed subgp of H.

Rem: Showed $T_e f$ injective & injective on geom pts $\iff$ scheme theoretic kernel is trivial

"Pf:" Last time, using closed orbit thm, showed $f(G)$ closed in H. WLOG, assume $H = f(G)$ so f bijective, $T_e f$ is bijective.

Strategy: can reduce to local question: $g: A \to B$ local k-algebras (local rings of f.g. k-alg's) A, B regular, g induces an iso on $A/m_A \xrightarrow{\sim} B/m_B$;

$m_A/m_A^2 \xrightarrow{\sim} m_B/m_B^2$

$\Rightarrow g$ an iso. (e.g. $g: A \to B$ iso $\iff \hat{g}: \hat{A} \to \hat{B}$ an iso. & "play the inverse fcn thm game")

$\hat{A} \simeq k[[x_1, \dots, x_n]] \hookrightarrow \hat{B} = k[[y_1, \dots, y_n]]$

$\square$

Normalizers, Centralizers, Transporters

$G \circlearrowright X \supseteq S$ closed

"$N_G(S) = \{g \in G \mid gS \subseteq S\}$"

"$C_G(S) = \{g \in G \mid gs = s \text{ all } s \in S\}$"

$T \subseteq X$ closed $\quad T_G(S,T) = \{g \in G \mid gS \subseteq T\}$

$$N_G(S) = T_G(S,S)$$

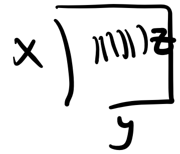
Guess $\quad N_G(S)(R) = \{g \in G(R) \mid gs \in S(R) \text{ all } s \in S(R)\}$

"correction" $\quad N_G(S)(R) = \{g \in G(R) \mid \operatorname{Spec} R \times S \xrightarrow{g \times \mathrm{id}_S} G \times S \xrightarrow{a} X$ factors through $S \hookrightarrow X\}$"

$$\operatorname{Spec} R \xrightarrow{g} G$$

$$\begin{tikzcd}
\operatorname{Spec} R \times S \arrow[r] \arrow[dr, "\exists"'] & G \times S \arrow[r] & X \\
& S \arrow[ur, hook] &
\end{tikzcd}$$

Def if X & Y schemes$/k$ $\quad Z \subseteq X \times Y$ closed



$\cap_Y Z\,(R) = \{x \in X(R) \mid \operatorname{Spec} R \times Y \to X \times Y$ factors through $Z \hookrightarrow X \times Y\}$

$(\cap_{y \in Y} Z_y)$

$$\begin{tikzcd}
\cap_Y Z \times Y \arrow[rr] \arrow[dr] & & X \times Y \\
& Z \arrow[ur] &
\end{tikzcd}$$

Ex Can show, equivalently

$$\cap_y Z(R) = \{x \in X(R) \mid \forall A/R \text{ R-alg.}, y \in Y(A)\ (x_A, y) \in Z(A)\}$$

$G \curvearrowright X \supseteq S$ closed

$$\begin{tikzcd} E \arrow[r] & G \times S \arrow[r, shift left, "gs"] \arrow[r, shift right, "s"'] & X \end{tikzcd}$$

E " $= \{(g,s) \mid gs = s\}$" $\quad (g,s)$

Def $C_G(S) = \cap_S E$

Ex: $C_G(S)(R) = \{g \in G(R) \mid \forall A/R\ s \in S(A),\ g_A \cdot s = s\}$

Normalizer:

$s = s_1$

$$\begin{tikzcd} F \arrow[r] \arrow[d] & S \times S \arrow[d] \\ G \times S \arrow[r] & X \times S \end{tikzcd} \qquad \begin{tikzcd} F_s \arrow[r, hook] \arrow[d] & F \arrow[d] \\ s \arrow[r, hook] & S \end{tikzcd}$$

$F = \{(g,s) \mid gs \in S\}$

$F_s = \{g \in G \mid gs \in S\}$

$g, s \longmapsto (gs, s)$

Def $N_G S = \cap_S F$

Ex: $N_G(S)(R) = \{g \in G(R) \mid \forall A/R,\ s \in S(A),\ g_A \cdot s \in S(A) \subseteq X(A)\}$

Def $\Omega_y Z$ $f \in \mathrm{Hom}(V, \Omega_y Z)$ is a map $V \to X$ s.t.

$Z \subseteq X \times Y$ closed

$$\begin{tikzcd} V \times Y \arrow[rr] \arrow[dr] & & X \times Y \\ & Z \arrow[ur] & \end{tikzcd}$$

Now we "know"

G affine k-gp variety, then $G \hookrightarrow GL_n$ closed subgp.
(smooth)

if $g \in GL_n(k)$ $T \in \mathrm{Mat}_n(k)$

say $k = \bar{k}$, after some basis chge, can make

$$T = \begin{bmatrix} \lambda_1 \; 1 & & \\ & \ddots \; \lambda_n & \\ 0 & & \ddots \end{bmatrix} \text{ Jordan form.}$$

$$= \underbrace{\begin{bmatrix} \lambda_1 \ddots \lambda_1 & & 0 \\ 0 & \lambda_2 & \ddots \end{bmatrix}}_{T_s} + \underbrace{\begin{bmatrix} 0 \; 1 \; 0 & & \\ & 0 \; 0 \; 1 & \\ & & \end{bmatrix}}_{T_n}$$

this decomp. is canonical

Def: $T \in M_n(k)$ is semisimple (if $k[T]$ is semisimple)
if whenever $W \subseteq k^n$ is T-stable
$\exists$ complementary $U \subseteq k^n$ T-stable w/ $W \times U \cong k^n$

this is equiv. to $k[T]$ being a semisimple ring.

$k[T] = k[x]/\min_T$ where $\min_T$ is a prod. of irred.'s w/ mult. 1

ex: $k = \mathbb{F}_p(z)$ may have $\min_T = X^p - z$ $n = p$

$\underset{L}{\underset{\shortparallel}{k[T]}}$ a field $\underset{L}{\underset{\shortparallel}{k^p}} \circlearrowleft L$

$\rightsquigarrow T_L \subseteq M_p(L)$

$\min_{T_L} = (X - \alpha)^p$
$\alpha^p = z$

$$\begin{bmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \alpha \end{bmatrix}$$

Next time (monday)
will show if k is perfect
then have canonical decompositions for a matrix T

$T = T_s + T_n$ T_s semisimple
T_n nilpotent
(proceed by $k \to \bar{k}$)

e.g. if $T \in GL_n$ $\quad T = T_s T_u$

T_u = unipotent

i.e. $I + N$ $\quad I = 1$ $\quad N$ = nilpotent

if $G \hookrightarrow GL_n$

will show for $g \in G(k)$

$g = g_s g_u$ $\quad g_s, g_u \in GL_n$ decomp.

then $g_s, g_u \in G(k)$

and is indep. of embedding

$G \hookrightarrow GL_n$.