Previously:

If V is a G-representation (functorial)

$G(R) \circlearrowright V_R$ functorially in R

i.e. nat transf. between functors G & GL_V

$R \mapsto G(R)$ $\quad R \mapsto \mathrm{Aut}_R(V_R)$

$G \times \mathbb{A}(V) \to \mathbb{A}(V)$

$$G(R) \times \underset{\underset{\mathbb{A}(V)(R)}{\cap}}{V_R} \to V_R$$

action of G on $\mathbb{A}(V) = \mathrm{Sym}^{\cdot}(V^*)$

"linear"

coord rings $\}$

$$S^{\cdot}(V^*) \to k[G] \otimes S^{\cdot}(V^*)$$

$$\cup \qquad\qquad \cup$$

$$V^* \to k[G] \otimes V^* \text{ comodule}$$

Also can always get V itself as a comodule

$V \to k[G] \otimes V$ $\quad$ observe: given any R pt. of G

$g \in G(R)$

$$V_R \xrightarrow[g]{} V_R$$

$$R \otimes V \longrightarrow R \otimes V$$

$g =$ "id" $\in G(k[G])$

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend right, "\text{coaction} = c"'] & k[G] \otimes V \arrow[r, "g"] & k[G] \otimes V \end{tikzcd}$$

if $h \in G(R)$ any point $\quad V_R \xrightarrow{h} V_R$ defined by c

via:

$$\begin{tikzcd} k[G] \arrow[r, "g"] \arrow[rr, bend right, "g_R"'] & S \arrow[r, "\psi"] & R \end{tikzcd}, \quad g \in G(S)$$

$$\begin{tikzcd} G(S) \times V_S \arrow[r] \arrow[d, "G(\psi)\times\psi"'] & V_S \arrow[d, "\psi"] \\ G(R) \times V_R \arrow[r] & V_R \end{tikzcd} \qquad \begin{tikzcd} V_S \arrow[r, "g"] \arrow[d, "\psi"'] & V_S \arrow[d, "\psi"] \\ V_R \arrow[r, "g_R"'] & V_R \end{tikzcd}$$

$S = k[G] \qquad h \in G(R) = \mathrm{Hom}(k[G], R)$ "ψ"

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend left, "c"] & k[G]\otimes V \arrow[r, "g"] \arrow[d, "h\otimes id_V"'] & k[G]\otimes V \arrow[d, "h\otimes id_V"] \\ V \arrow[r] & R\otimes V \arrow[r, "g_R = h"'] & R\otimes V \end{tikzcd}$$

id

$\hat{=}$

$g \in G(k[G]) \qquad g_R$ induced by $h: k[G] \to R$

then $\boxed{g_R = h}$

$$\begin{tikzcd} k[G] \arrow[r, "g"', "id"] \arrow[dr, "h"'] & S \text{ "}k[G]\text{"} \arrow[d, "\psi = h"] \\ & R \end{tikzcd}$$

$\psi \circ g = g_R$

<u>Exercise</u>: if $g \in G(k)$, $g \cdot v$ given by

if $c(v) = \sum f_i \otimes v_i \qquad g \cdot v = \sum f_i(g) \cdot v_i$

$\quad \in k[G]\otimes V$

"Construction of all representations" of a LAG G

(following Waterhouse 3.5)

If $A = k[G]$ then have natural rep on A (as a base)

Lem if V is any G-rep, then V is a subrep. of A^n some n.

($n = \dim V$)

Pf: Consider $V \otimes A$ as a comod via A-side

$A \xrightarrow{\Delta} A \otimes A$ $\qquad c : V \longrightarrow A \otimes V$

$V \otimes A \xrightarrow[\tilde{c}]{id_V \otimes \Delta} V \otimes A \otimes A$

Observe: $V \otimes A$ as above is $\cong$ to $A^{\dim V}$

but we have a map $V \xrightarrow{c} V \otimes A$ and this is a comodule map!

Check comodule map, want

$$\begin{tikzcd}
V \arrow[r, "c"] \arrow[d, "c"'] & V \otimes A \arrow[d, "c \otimes id_A"] \\
V \otimes A \arrow[r, "\tilde{c} = id_V \otimes \Delta"'] & V \otimes A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "a_V"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & V \otimes A \arrow[d, "\varphi \otimes id_A"] \\
W \arrow[r, "a_W"'] & W \otimes A
\end{tikzcd}$$

commutes by coassociativity of the action. $\square$

Prop if $G \hookrightarrow GL(V)$ any faithful f.d.im'l rep,

then any other f.d.im'l rep $G \to GL(W)$ is constructed by operations of $\oplus$, duals, $\otimes$, subrep, quotients.

Remark: $\oplus$, subrep, quotient straightforward.

if $\rho_i : G \to GL(V_i)$ $i=1,2$

$$\begin{tikzcd} G \arrow[r, "\rho_1 \oplus \rho_2"] \arrow[dr, "\rho_1 \times \rho_2"'] & GL(V_1 \oplus V_2) \\ & GL(V_1) \times GL(V_2) \arrow[u, hook] \end{tikzcd}$$

$$GL(V_1) \hookrightarrow GL(V_1 \oplus V_2) \longleftrightarrow \begin{bmatrix} * & \\ & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \\ & * \end{bmatrix}$$

$\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$

$g \longmapsto \rho_1(g) \otimes \rho_2(g)$

given T_1, T_2

$T_i \in End(V_i)$

$(T_1 \otimes T_2)(v_1 \otimes v_2)$

$\equiv T_1(v_1) \otimes T_2(v_2)$

$\rho : G \to GL(V)$ V f.dim'l

$\rho^* : G \to GL(V^*)$

$\rho^*(g)(f) = f \circ g^{-1}$

matrices $V = k^n$ $V^* = k^n$ dual basis

dual corresp. to (comp. w/)

$GL(V) \to GL(V^*)$

$T \longmapsto (T^t)^{-1}$

Pf (sketch)

Given $G \hookrightarrow GL(V)$ $\quad A = B/I$ some I

$$\begin{array}{ccc} G & \hookrightarrow & GL(V) \\ \| & & \| \\ \operatorname{Spec} A & & \operatorname{Spec} B \end{array}$$

$B = k[x_{ij}][\det(x_{ij})^{-1}]$

$= k[x_{ij}][d] \big/ d \cdot \det(x_{ij}) - 1$

enough to show that

every subrep of A (as a comod)

is obtained from V via $\otimes$, duals, subs, ...

observe: $G \hookrightarrow GL(V) \circlearrowright A$ that B is a rep of G

and $I \subseteq B$ is a sub rep.

$\left(\begin{array}{l} \text{if } g \in G(k) \quad f \in A \text{ is in } I \iff f(\text{elmts of } G) = 0 \\ \qquad \text{w.t.s } (g \cdot f)(h) = 0 \text{ for } h \in G \\ \qquad\qquad \overset{\|}{f(g^{-1}h)} \quad g^{-1}, h \in G \Rightarrow g^{-1}h \in G \\ \qquad\qquad\quad \overset{\shortparallel}{0} \end{array} \right)$

ets that f.d subs of B are obtained as above.

$B = k[x_{ij}, d] / .$ $\quad V = \langle e_i \rangle$ basis corresp to x_{ij} fcns.

$$c(e_j) = \sum x_{ij} \otimes e_i$$

subspace of B gen by $\langle x_{ij} \rangle_i$ $\langle x_{ij} \rangle_j$

$(\qquad)(\,) \qquad (\qquad)(\qquad)$

i.e. $\langle x_{ij} \rangle \subseteq B$ is V^n as a sub rep.

$$G \to GL(V) \circlearrowright B \supseteq \langle x_{ij} \rangle \simeq V^n$$
$$\circlearrowright V$$

quad expressions in x_{ij}

$\langle x_{12}x_{34}, \ldots \rangle$

quot. of rep $V^n \otimes V^n$

$x_{12} \otimes x_{34} \longrightarrow x_{12}x_{34}$

any poly expression in x_{ij}'s is sim. quot

of $V^n \otimes \cdots \otimes V^n$

sums of

$k[x_{ij}] \subseteq B$ $\quad$ $\langle \det x_{ij} \rangle \subseteq B$

$\cup$

d

is a 1-dim'l subrep of $GL(V)$

& hence of G

$$GL(V) \longrightarrow GL(\Lambda^n V)$$
$$g \longmapsto (v_1 \wedge \cdots \wedge v_n \mapsto g(v_1) \wedge \cdots \wedge g(v_n))$$

$GL(V) \in \langle d \rangle$ via $\langle \det(x_{ij})^{\times} \rangle$

$\det(x_{ij}) \subseteq (V^n)^{\otimes n}$

similarly $\langle d^r \rangle$ is $\left(\langle \det(x_{ij}) \rangle^{\times} \right)^{\otimes n}$