Suppose $g \in \mathrm{End}_k(V)$ lin. trans.

$k[g] = k[x]/m_g$

we say g is semisimple if $k[g]$ is a semisimple alg.

$\Updownarrow$

semiprimitive & artinian

$\Updownarrow$

$k[g] = R$ $R = \oplus S_i^{n_i}$ of simple mods over itself

"S_i's are same"

$\Updownarrow$

$\forall M/R$, $M \simeq \oplus S_i$ S_i simple

recall: if R/k f.d. k-alg then R semisimple

$\iff R \simeq \prod M_{n_i}(R_i)$ R_i/k div. alg.

R comm. for us $\Rightarrow R \simeq \prod E_i$ E_i/k finite field ext.

$k[g]$ ssimple $\Rightarrow$

where $k[g] \curvearrowright V$ $V \simeq \oplus V_i$ $V_i/k[g]$ simple as a $k[g]$-mod.

$k[g] = \prod E_i$

Quick Review of semisimplicity

Def A ring R is (left) semisimple if $R \simeq \bigoplus S_i$ S_i is a simple (left) R-module

Lem: if R is semisimple & M a (left) R-mod $\Rightarrow$
$M \simeq \bigoplus$ simples, each simple arises in decomp. of M.

Def M an R-module is semisimple if $M \simeq \bigoplus$ simples.

Ex: k field. k simple as a module over itself.
$\Rightarrow V$ a k-mod $\Rightarrow V \simeq \bigoplus S_i$ S_i simple arising in
$S_i \simeq k$

$$V \simeq \bigoplus_{i \in \Lambda}^{\wedge} k$$

Wedderburn: if R a f.d. semisimple k-alg $\Rightarrow$
$R \simeq M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$
D_i a finite dim'l division alg$/k$

if R comm. then D_i = fields $n_i = 1$
$\Rightarrow R$ f.dim'l semisimple k-alg is of form $R \simeq E_1 \times \cdots \times E_r$
E_i/k field ext.

Def an R-module M is simple (aka irreducible)
if only submodules are $0, M$.

Def an R-module M is decomposable if $M \simeq N_1 \oplus N_2$, $N_i \neq 0$. If M is not decomp it is called indecomposable.

Lem: R semisimple $\Leftrightarrow$ every indecomposable module is simple.

$g \in \mathrm{End}_k V$ $\quad k[g]$

Def g is separable if $k[g]$ is an étale k-algebra.

Reminder: R/k is étale if $R \simeq E_1 \times \dots \times E_r$, E_i/k is a separable field ext.

(slightly stronger than semisimple)

k perfect $\Rightarrow$ same

Prop (Wedderburn)

if R is finite dim'l $/k$, $J = J(R)$ then

$$R \simeq S \oplus J \text{ as } k\text{-vectorspaces}$$

where S is a semisimple k-alg.

$R \to R/J \leftarrow$ semisimple $\qquad$ also note: J nilpotent.

For us, if $g \in \mathrm{End}_k(V)$

$$k[g] \simeq S \oplus J \Rightarrow g = g_{ss} + g_n$$

where g_{ss} is semisimple & g_n is nilpotent.

S (& J) gen. by image of g

g_{ss} = image of g in S $\quad$ g_n = im. of g in J

$$k[g_{ss}] \simeq S$$

$\Rightarrow$ lnr trans g_{ss} is semisimple & g_n is nilpotent.

ex: $g = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\qquad$ $k[g] \simeq k[x]/(x-2)^2 \overset{\text{vspc}}{\simeq} k \oplus kx$

$$\simeq \underset{S}{k} \oplus \underset{J}{k(x-2)}$$

$$g \leftrightarrow x = \underset{g_{ss}}{\overset{\shortparallel}{2}} + \underset{g_n}{\overset{\shortparallel}{(x-2)}} \qquad \left(x = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right)$$

$$x-2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$g \longleftrightarrow \begin{bmatrix} 2 & 1 & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & 4 \end{bmatrix} = \begin{bmatrix} 2 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix}$$

$$\qquad\qquad\qquad g_{ss} \qquad\qquad g_n$$

<u>Remark</u>: if $g \in GL(V)$ then $g_{ss} \in GL(V)$

$\Rightarrow$ can define $g_u = g g_{ss}^{-1} = 1 + g_n g_{ss}^{-1}$

$$g = g_{ss} + g_n$$

<u>note</u>: $g_{ss}, g_n, g \in k[g]$ comm. alg $\Rightarrow$ they all commute w/ each other.

g_u = unipotent part of g

g_{ss} = ss. part $\quad g_n$ = nilpotent part

$$g = g_{ss} + g_n \qquad g = g_{ss} g_u = g_u g_{ss}$$

"Chevalley - Jordan decomp."

<u>Prop</u>: If g_{ss} is <u>separable</u> (if $k[g_{ss}]$ sep)

then the decomp $g = g_{ss} + g_n$ (or $g = g_{ss} g_u$)

is **unique**.

& preserved by rep's $GL(V) \xrightarrow{\varphi} GL(W)$

$$g = g_{ss} g_u \rightsquigarrow \varphi(g) = \varphi(g_{ss})\varphi(g_u)$$

$$\varphi(g_{ss}) = \varphi(g)_{ss}$$

$$\varphi(g_u) = \varphi(g)_u$$

ss = semisimple sep = sep. s = both coincide
not distinguish

Theorem: if G is a lin. alg. gp/k, k perfect

$G \overset{\varphi}{\hookrightarrow} GL(V)$ closed, $g \in G(k)$ then $\exists!\, g_s, g_u \in G(k)$

s.t. $\underset{\varphi(g)_{sep}}{\varphi(g)_{ss}} = \varphi(g_s)$

$\varphi(g)_u = \varphi(g_u)$

Idea of proof

$$\begin{array}{ccc} G & \hookrightarrow & GL(V) \\ \| & & \| \\ \operatorname{Spec} A & & \operatorname{Spec} B \end{array}$$

$A = B/I$ (φ = suppressed inclusion)

Goal: if $g \in G(k)$ then $g_{ss} \in G(k)$

via: if $f \in I$ then $f(g_{ss}) = 0$

$g \circlearrowright$ (finite dim'l subsets of) B

g_{ss}, g_u parts are preserved.

Quick moral argument

Show: $gI \subseteq I$ same as $g \in G(k)$

if $f \in I$ $g \cdot f \in I$? $(g \cdot f)(h) = f(\underbrace{g^{-1}h}_{\in G}) = 0$

$h \in G$

conversely if $gI \subseteq I$ $f \in I$ why is $f(g) = 0$?

$f(g \cdot e) = (\underset{I}{g^{-1} \cdot f})(\underset{G}{e})$

$= 0$

WTS: $g_{ss} I \subseteq I$

$k[g] \subset B' \subseteq B$

f.d.

$g = g_{ss} \cdot g_u$ same decomp as original

$g_{ss} \in k[g]$ so lin. comb. of powers of g

$gI \subseteq I$

$\Rightarrow k[g] I \subseteq I$

$g_{ss} I \subseteq I$ ✓ ☺