From last time:

Given $G \underset{\rho}{\hookrightarrow} GL(V)$ representation

want to show: if $\rho(g) = \rho(g)_{ss}\rho(g)_u$ then

$\rho(g)_{ss} = \rho(g_{ss})$

$\rho(g)_u = \rho(g_u)$

for some $g_{ss} \in G$

$g_u \in G$

$$\boxed{\begin{aligned} A &= k[G] \\ B &= k[GL(V)] \\ A &= B/I \end{aligned}}$$

Idea of argument:

Lemma: $h \in GL(V)$ is of the form $\rho(h')$

$\iff$

$hI \subseteq I$ (in the representation $GL(V) \circlearrowright B$)

$\rho(g)_u, \rho(g)_{ss} \in k[\rho(g)] \subseteq End(V)$

$\rho(g)_{ss} I \subseteq k[\rho(g)] I \subseteq I$

$\rho(g)_u I \quad \dots \quad \subseteq I \quad \square$

Prop: if k perfect (so semisimple = étale)
then $g = g_s g_u$ unique (coincides from any rep)

Pf: suppose $\rho : G \hookrightarrow GL(V)$ faithful rep, write $g = g_s g_u$
from this $\rho(g_s) = \rho(g)_{ss}$ $\rho(g_u) = \rho(g)_u$

if W another rep want same decomp.

Consider $W \subseteq V$ sub $V \twoheadrightarrow W$ or $W = V_1 \otimes V_2$

or $W = V^*$

For ss part: $k[g] \to \mathrm{End}(V)$

$$\begin{array}{l} k[g] \to \mathrm{End}(V) \\ \cup \\ k[g]_{ss} \nearrow k[g_{ss}]/I^{\,ss.} \\ \parallel \qquad\qquad \downarrow \\ k[g_{ss}] \to \mathrm{End}(W) \\ \\ k[g_n] \nearrow \text{nilp} \end{array}$$

if $W \subseteq V$ or $W \twoheadleftarrow V$

Remark: if k is perfect then $k[g]$ is semisimple $\Leftrightarrow$ étale

$\Longleftrightarrow$

$k[g] \otimes_k \bar{k} \cong \bar{k} \times \dots \times \bar{k}$

$$\frac{k[x]}{f} \otimes_k \bar{k} \simeq \frac{\bar{k}[x]}{f} \simeq \prod \frac{\bar{k}[x]}{(x-\alpha_i)} \simeq \bar{k}^n$$

irred, k perfect $\Rightarrow f$ separable, dist. roots

Note if L/k not sep $\Rightarrow L \otimes_k \bar{k} \not\cong \bar{k}^n$ any n.

$$\cong \prod \frac{\bar{k}[x]}{(x-\alpha_i)^{n_i}} \simeq \prod \bar{k}[\varepsilon]/_{\varepsilon^{n_i}}$$

$x_{new} = x - \alpha_i$

g separable $\iff k[g]$ separable $\iff \bar{k}[g] \simeq \bar{k}^n \simeq \prod \frac{\bar{k}[x]}{x-\alpha_i}$

$\bar{k}[g] = k[g] \otimes_k \bar{k}$

$\iff g$ is diagonalizable over $\bar{k}$

similarly unipotent means all evals are 1 over $\bar{k}$

if W another rep want same decomp.

Consider $W \subseteq V$ sub $\quad V \twoheadrightarrow W \quad$ or $\quad W = V_1 \otimes V_2$

or $W = V^*$

if g_s is diag. over $\bar{k}$ in action on V

then it is also diag. on action on $W \subseteq V$

or V/W

$k[g_s]$ ss. $\Rightarrow$ every module is a sum of simples in $k[g_s]$

$\Rightarrow$ simples are summands of $k[g_s]$

$\Rightarrow$ simples are projective

$\Rightarrow$ every module is proj

$\Rightarrow \quad W \subseteq V$ submod $\Rightarrow V \simeq W \oplus U$

$g \curvearrowright V = W \oplus U$

$$\begin{array}{c} W \\ \begin{bmatrix} \text{Jordan} & \\ & \text{Jordan} \end{bmatrix} \\ U \end{array}$$

dual ens $T \longrightarrow (T^t)^{-1}$

preserves diagonalizable.

$\underset{\text{diag}}{T} \otimes \underset{\text{diag.}}{S}$ = diag. via Kronecker prod.

$\square$.

if T is sep. & U is unip.

and $T = U$ then $T = U = 1$

$\Rightarrow$ decomp is unique.

$g = g_s g_u = \overset{\text{sep}}{T}\,\overset{\text{unip.}}{U}$ & T, U comm. w/ g

$T^{-1} g_s = U g_u^{-1} \Longrightarrow T^{-1} g_s = 1 = U g_u^{-1}$

$\Rightarrow T = g_s$ & $U = g_u$

if T_1, T_2 sep & commute

$\Rightarrow T_1 T_2$ is sep

if U_1, U_2 unip & comm

$\Rightarrow U_1 U_2$ is unip

$k[T_1] \curvearrowright V \curvearrowleft k[T_2]$

$\Rightarrow k[T_1, T_2] \curvearrowright V$

$k[T_1] \otimes k[T_2]$

sep.

$\mathrm{Sep} \otimes \mathrm{Sep} = \mathrm{Sep}$

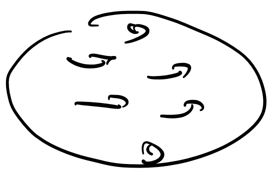
$$(R_1 \otimes_k R_2) \otimes_k \bar{k} = (R_1 \otimes_k \bar{k}) \otimes_{\bar{k}} (R_2 \otimes_k \bar{k})$$
$$= \bar{k}^n \otimes_{\bar{k}} \bar{k}^m = \bar{k}^{nm}$$

$u_1 u_2 - 1$ nilp?

$u_1(u_2 - 1) + (u_1 - 1)$ is nilp. ✓

Linear alg. gp mythology

Lie gps via vector fields of sym.



compact Lie gps ⇝ classification of simple

$T \to GL_n \quad T \subset G \qquad \overset{\mathbb{Z}^n}{(S^1)^n} \cong T \le G \subset V$

$\mathbb{C}^*$ conj

max'l tori

$S^1 \hookrightarrow GL_1$

dim $S^1 \subseteq \mathbb{C}^\times$

$(\mathbb{C}^\times)^n \cong T \le G$ LAG

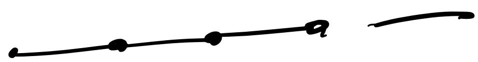
alg. tori

Classification of Lie gps, finite gps, Linear alg gps

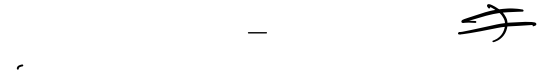
simples mostly come from v.spaces w/ form

bilin form
sym. or antisymm
or Hermitian

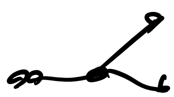
A_n — v.space (w/ Herm fm)
B_n — odd dim'l v.space w/ bilin fm
C_n — skew forms
D_n — even dim'l v.spaces w/ bilin fms
$E_{6,7,8}$ "Brown"
F_4 — Albert algs
G_2 — octonion /Cayley algs

A_n 

B 

C 

D 

D_4  

"trialitarian algs"